Theoretical aspects of long-term evaluation in environmental economics

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Christian P. Traeger

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Contents

Notation in parts II and III

Symbol	Explanation	Page
$\succeq$	weak preference relation	72, 77, 79
$\succ$	strong preference relation	72
$\sim$	indifference relation	72
$\succeq_t$	preference relation in period t on P_t	110
$\succeq\vert_{\mathsf{X}}$	restriction of $\succeq$ to the set of certain consumption paths	80
$\equiv$	defining equality	
A	$\mathsf{A} = \{\mathfrak{a} : \mathbb{R} \to \mathbb{R} : \mathfrak{a}(z) = a\,z + b\,,\ a, b \in \mathbb{R}, a \neq 0\}$,	
	group of affine transformations	74
A^+	$\mathsf{A}^+ = \{\mathfrak{a}^+ : \mathbb{R} \to \mathbb{R} : \mathfrak{a}^+(z) = a\,z + b\,,\ a, b \in \mathbb{R}, a > 0\}$	74
A^a	$\mathsf{A}^a = \{\mathfrak{a}^a : \mathbb{R} \to \mathbb{R} : \mathfrak{a}^a(z) = az + b, b \in \mathbb{R}\}$	113
AIRA	measure of absolute intertemporal risk aversion	100
AIRA_t	measure of absolute intertemporal risk aversion in period t	123
$B_{\succeq}$	set of Bernoulli utility functions for $\succeq$,	
	$\{u \in \mathcal{C}^0(X) : [x]_1 \succeq [x']_1 \Leftrightarrow u(x) \geq u(x')\,\forall\, x, x' \in X\}$	73, 77, 80, 130
$B_{\succeq_t}$	set of Bernoulli utility functions, non-stationary setting	
	$\{u_t \in \mathcal{C}^0(X) : [x]_t \succeq_t [x']_t \Leftrightarrow u_t(x) \geq u_t(x')\,\forall\, x, x' \in X\}$	111
$\mathcal{C}^0(X)$	space of all continuous functions from X to $\mathbb{R}$	66
$\Delta(Y)$	space of Borel probability measures on Y	66
ΔG_t	$\Delta G_t = \overline{G}_t - \underline{G}_t$	111
exp	exponential function	
fg^{-1}	$f \circ g^{-1}$, composition	
$\hat{f}$	$\hat{f} = \{\mathfrak{a}f : \mathfrak{a} \in \mathsf{A}\}$	91
$\hat{f}^{-1}$	$\hat{f}^{-1} = \{f^{-1}\mathfrak{a} : \mathfrak{a} \in \mathsf{A}\}$	91
$\underline{G}$	$\underline{G} = g(\underline{U})$	76
$\overline{G}$	$\overline{G} = g(\overline{U})$	76
G	$G = [\underline{G}, \overline{G}]$	76
$\underline{G}_t$	$\underline{G}_t = g_t(\underline{U}_t)$	111
$\overline{G}_t$	$\overline{G}_t = g_t(\overline{U}_t)$	111
G_t	$G_t = [\underline{G}_t, \overline{G}_t]$	111
Γ	$\Gamma = (\underline{G}, \overline{G})$	76
Γ_t	$\Gamma_t = (\underline{G}_t, \overline{G}_t)$	111

Symbol	Explanation	Page
id	identity	
$\lambda x + (1-\lambda)x'$	lottery over outcomes x and x' with respective probabilities λ and $1-\lambda$, $\lambda x + (1-\lambda)x' \in P$	66
ln	natural logarithm	
$\mathcal{M}^f$	uncertainty aggregation rule, $\mathcal{M}^f : \Delta(Y) \times \mathcal{C}^0(Y) \to \mathbb{R}$ with $\mathcal{M}^f(p,u) = f^{-1}\left[\int_Y f \circ u \, dp\right]$, $f : \mathbb{R} \to \mathbb{R}$ strictly monotonic and continuous	67
$\mathcal{M}^\alpha$	shorthand for $\mathcal{M}^{\text{id}^\alpha}(p,u) = \left[\int_Y u^\alpha dp\right]^{\frac{1}{\alpha}}$	68
$\mathcal{M}^0$	shorthand for $\lim_{\alpha \to 0} \mathcal{M}^\alpha(p,u) = \exp\left[\int_Y \ln(u)\, dp\right]$	68
$\mathcal{M}^{f_t}$	uncertainty aggregation rule in period t	111
$\mathcal{N}^g$	intertemporal aggregation rule, stationary, no discounting $\mathcal{N}^g : U \times U \to \mathbb{R}$ with $\mathcal{N}^g(\cdot,\cdot) = g^{-1}\left[\frac{1}{2}\, g(\cdot) + \frac{1}{2}\, g(\cdot)\right]$	78
$\mathcal{N}^{g_t, g_{t+1}}$	intertemporal aggregation rule, non-stationary	112
P	space of Borel probability measures on X	66
p	uncertain outcomes or lotteries, $p \in P$	66
P_t	general choice space in period t	110
p_t	period t lottery, $p_t \in P_t$	110
p_t^{X}	reduced probability measure on certain consumption paths, $p_t^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$	165
$\mathbb{R}_+$	$\mathbb{R}_+ = \{z \in \mathbb{R} : z \geq 0\}$	67
$\mathbb{R}_{++}$	$\mathbb{R}_{++} = \{z \in \mathbb{R} : z > 0\}$	67
range	range of a function	
RIRA	measure of relative intertemporal risk aversion	99
RIRA_t	measure of relative intertemporal risk aversion in period t	123
RRA	Arrow-Pratt-measure of relative risk aversion	86
θ_t	normalization constant, non-stationary representation	112
ϑ_t	normalization constant, non-stationary representation	112
u	Bernoulli utility function, $u \in \mathcal{C}^0(X)$	66, 73, 77
$\underline{U}$	$\min_{x \in X} u(x)$	66
$\overline{U}$	$\max_{x \in X} u(x)$	66
U	$\text{range}(u) = [\underline{U}, \overline{U}]$	66
$\underline{U}_t$	$\min_{x \in X} u_t(x)$	111
$\overline{U}_t$	$\max_{x \in X} u_t(x)$	111
U_t	$[\underline{U}_t, \overline{U}_t] = \text{range}(u_t)$	111
u^{welf}	welfare, certainty additive Bernoulli utility function	96

Symbol	Explanation	Page
X	connected compact metric space of outcomes	66
x	consumption levels or (certain) outcomes, $x \in X$	66
X^t	space of consumption paths starting in period t	76
x	consumption path	76
x^t	consumption path from period t to period T, $\mathsf{x}^t \in \mathsf{X}^t$	76
x^t_τ	period τ entry of consumption path x^t	76
$[x]_t$	consumption path $[x]_t \in \mathsf{X}^t$ giving consumption x in period t and some fixed $x^0 \in X$ for all subsequent periods	76
$\bar{\mathsf{x}}^t$	$\bar{\mathsf{x}}^t = (\bar{x}, \bar{x}, ..., \bar{x})$, certain constant consumption path	133
$(\mathsf{x}_{-i}, \mathsf{x}'_i)$	$(\mathsf{x}_t, ..., \mathsf{x}_{i-1}, \mathsf{x}'_i, \mathsf{x}_{i+1}, ..., x_T) \in \mathsf{X}^\tau$	119
$\tilde{X}_t$	degenerate choice space in period t	110
$\tilde{x}_t$	degenerate period t lottery	110
Y	connected compact metric space, used for general definitions	

Notational Remark:

Sections within a chapter are referred to as *section x.y*, while sections in another chapter are referenced as *chapter x.y*.

Chapter 1

Introduction

1.1 Motivation: Environmental Change

Economic activity has always altered societies' physical and biological environment. These human-induced changes have grown with population density and technical capability. In recent times humankind's impact on the natural environment has not only changed in quality and quantity, but also the time scale on which its actions affect the physical and biological surroundings has increased significantly. I want to point out this phenomenon at the example of global climate change. It is probably one of the most serious human induced alterations of the natural environment discovered so far. The background for this so called 'global warming problem' is that the atmosphere of the earth naturally traps incoming solar energy. This process results in a higher average temperature on the planet's surface. The effect is mainly caused by water vapor and different atmospheric gases. Due to the resemblance of the radiative forcing effect of these gases to that of a greenhouse, these gases are typically classified as greenhouse gases. Without the natural greenhouse effect, today's average surface temperature of the planet would be approximately 33°C colder (Roedl 2000). As a result of human economic activity, accumulation of these greenhouse gases in the atmosphere has increased over the past centuries and, in particular, over the last decades. The most important factor in this human induced accumulation is the increase in the atmospheric carbon dioxide concentration, mainly caused by fossil fuel burning and deforestation. Another example for an even more efficient greenhouse gas, which appears, however, in lower concentrations in the atmosphere, is methane. Methane is also emitted in fossil fuel burning but, most importantly, as well in cattle and rice agriculture and from landfills.[1] Such an increase in greenhouse gas concentration causes an (additional) increase in global surface temperature, as well as a change in other climatic determinants.

Considering different scenarios for future economic activity, the International Panel on Climate Change (IPCC) predicts with respect to the baseline of 1990 an increase in average global surface temperature between 1.4 to 5.8 °C by the end of the century, with a disproportional temperature increase on land areas (IPCC 2001*b*). While small amounts of climate change can proof beneficial for some regions and sectors, adverse effects are predicted to dominate and to increase with the magnitude of change. Adverse effects include sea level rise, threats to human health, decrease in ecological productivity and increased precipitation variability causing floods and droughts (IPCC 2001*e*, IPCC 2001*a*). Another large scale change of a climate determinant that could be triggered by global warming, is a shut-down of the northern arm of the Gulf Stream. Such a process would be irreversible, and bring a significant cooling and change in the precipitation pattern for Northwestern Europe.[2] Other examples of irreversible processes that can be caused by global warming include an irreversible disintegration of the West Antarctic ice sheet, resulting in a permanent rise of the sea level (IPCC 2001*e*, 77), and an acceleration of biodiversity loss (IPCC 2001*a*, 53). The irreversibility involved in these potential changes highlights the time scale in which current actions, and emissions within a couple of decades, can alter our natural environment. However, also without invoking irreversibility, the example of global warming points out that environmental change and feedback, caused by today's decisions, affect a long time horizon. For example, the thermal expansion of the oceans and melting of the ice will continue at least for several hundred years after a stabilization of greenhouse gas concentrations has been achieved (IPCC 2001*b*, 17). An additional characteristic of climate change is the high degree of uncertainty involved in most quantitative estimates concerning the effects of global warming and the amount of global warming needed to trigger other changes in the climate system (IPCC 2001*b*, 30 et seqq.).

While climate change alone would certainly be motivation enough for this study, a similar reasoning on time scale and uncertainty holds true for many other economy-environment interactions. Such examples include biodiversity loss, depletion of the ozone layer, potential impacts of genetically modified organisms, ecosystem impact of

[1] For an overview of the different causes of global warming compare with increasing detailedness IPCC (2001*d*), IPCC (2001*e*) and IPCC (2001*c*). A recent study by Keppler, Hamilton, Braß & Röckmann (2006) suggests that also the production of methane in intact plants and from detached leaves plays a more important role in the global methane budget than previously assumed.

[2] A recent study by Bryden, Longworth & Cunningham (2005) indicates that the volume of water carried by the Atlantic Conveyor Belt to Northwestern Europe has possibly dropped already by 30%.

invasive species, nuclear waste disposal, watershed protection and climate regulation functions of tropical rain forests, and soil degradation. There are three *stylized facts* that can be derived from these examples. First, there are important settings in which a careful consideration of the *long term* is essential for an adequate determination of costs and benefits going along with economic activities that cause changes to the physical and biological environment. Second, in the mentioned examples, benefits gained from most economic activities[3] are comparatively short-term as opposed to the long-term repercussions on welfare through the induced environmental change. This *time structure* makes it extremely important, to carefully think about weighing the future as opposed to the present in order to avoid a systematic bias between (short-term) benefits and (long-term) costs. And third, the magnitude of contemporary benefits from economic activity is generally much better *known* than the highly *uncertain* magnitude (or even character) of environmental changes and their repercussion on welfare in the long run. Therefore, a careful consideration concerning the evaluation of uncertainty associated with the long-term consequences of environmental change is a challenge of high priority. *Treating either the time structure or the uncertainty structure in the economic framework of balancing costs against benefits inattentively, is likely to bias policy recommendations in a systematic way in favor or against environmental interventions.*

1.2 Conceptual Background

In economic theory, welfare is usually characterized by a function, or a more general mathematical relation, that allows to compare different scenarios, and to evaluate which choice in a given set of possibilities is best. The process of evaluating a particular scenario or project is called *cost benefit analysis*. To this end, cost benefit analysis has to identify different determinants of welfare, and make them comparable at a given point of time. In such an analysis, the environment serves as a source of human welfare in a manifold way. First, this happens by immediate appreciation of environmental amenities such as clean air, scenic views, outdoor activities or appreciation of biodiversity. Second, our physical and biological surroundings give rise to more indirect service flows, like for example, the regulation of hydrological cycles or climate regulating functions. Third, our natural environment provides resources that enter into the economic production

[3]That is, fossil fuel burning, agriculture, deforestation and landfill storage mentioned above as some of the major causes for the increase of greenhouse gas concentrations, or the usage of CFC's or genetically modified organisms and again deforestation as major causes for the other mentioned examples.

process. All of these contributions to human welfare are captured under the concept of use value.[4] In addition to these use-values, one considers as well non-use (or intrinsic) values that generally arise from ethical, moral or spiritual considerations. Moreover, cost benefit analysis does not only have to compare different outcomes at a given point of time, but also has to make comparisons when costs and benefits occur at different points of time. As it was observed in the preceding section, a project that involves fossil fuel burning or deforestation might go along with immediate benefits, as well as an unfavorable alteration of the environment that feeds back into the welfare function in the long run. Therefore, an important determinant in a cost benefit analysis is the weight given to future sources of well-being as opposed to the current. These weights, assigned to different points of time, are usually characterized by means of a (social) discount rate.

The *social discount rate*, and thus, the weight given to future consumption and service streams, is usually composed of two factors. The first is the so called *rate of pure time preference*. It characterizes society's impatience or intrinsic devaluation of the future, and is typically an exogenous parameter to the economic theory of cost benefit analysis. While usually positive rates of pure time preference are employed, some share the point of view prominently expressed by Ramsey (1928, 543) that a pure rate of time preference is "ethically indefensible and arises merely from the weakness of the imagination". The second determinant of the social discount rate is incorporating evaluative effects of economic growth and decreasing marginal utility. More precisely, it is characterized by the growth rate which is weighted with a term that depicts society's desire to smooth consumption over time. This term is founded on a simple economic consideration relating the technological possibility of increasing produced consumption over time through investment, and the willingness to forego consumption in some period in order to have more in another. Dasgupta (2001, 183 et seqq.) works out that this consideration, in combination with a decline of long-term economic growth caused by global climate change, could increase the weight that should be given to future consumption and service streams. His reasoning implies that the optimal social discount rate would be falling over time. As constant discount rates go along with an exponential decline of weight given to the future, falling discount rates, which yield a more moderate decline of future-weight, are often regarded as favorable. However, applying such declining discount rates

[4]Another (indirect) use value which is often mentioned in this context is the so called option or quasi option value. It is introduced to capture the value of an uncertain potential use of natural resources in the future, when time is not considered explicitly. However, in a dynamic framework, as employed in this dissertation, these option values fall together with uncertain use values in the future (and possibly also with future non-use values as stated next).

in project evaluation can result in a continuous revision of the optimal project, due only to the passage of time and not to any real changes in the project or its circumstances.[5] This phenomenon has been termed *time inconsistency*. Most approaches that suggest declining discount rates on the basis of intergenerational justice or observed behavior, lead to such a time inconsistency. However, more recently, different approaches that rationalize time consistent declining discount rates under uncertainty have been developed (Weitzman 1998, Azfar 1999, Dasgupta & Maskin 2005). In relation to climate change, Nordhaus' (1993,1994) integrated assessment model for climate change and its critical discussions and alternative assessments have shown the importance of carefully quantifying the social discount rate for the derivation of an optimal greenhouse gas abatement path (see e.g. Toth 1995, Plambeck, Hope & Anderson 1997). In particular, Plambeck et al. (1997, 85) point out that a reduction of the pure rate of time preference from 3%, as assumed by Nordhaus (1993), to 0%, would result in an optimal abatement path that cuts emissions by 50% from the baseline to the year 2100, as opposed to 10% in the assessment of Nordhaus (1993).

The relation between current and future well-being is the central focus of a *sustainable development*.[6] With respect to the discussion on discount rates in the preceding paragraph, let me remark that Pezzey (2006) refers to discount rates that fall over time as 'sustainable discount rates'. Van den Bergh & Hofkes (1998, 11) describe the common denominator of sustainability theories as the acknowledgment of the "long-run mutual dependence of environmental quality and resource availability on the one hand, and economic development on the other hand". In this context two different notions of sustainability are generally distinguished. On the one hand, there is the *weak sustainability* paradigm which is mainly concerned with the preservation of a non-decreasing overall welfare. To this end, advocates of the notion of weak sustainability allow for a substitution between environmental and man-made capital. On the other hand, the paradigm of *strong sustainability* requires a non-declining value or physical amount of natural capital and its service flows. The latter demand of non-declining natural capital and service flows, is based on the viewpoint that substitution possibilities between man-made goods and natural resources and service flows are either limited or ethically indefensible. While

[5] This problem does not apply to the mentioned reasoning set forth by Dasgupta (2001, 183 et seqq.), where the falling discount rates are implied by a change in future economic development.

[6] The report of the World Commission on Environment and Development (Brundtland report) defines: "Sustainable development is development that meets the needs of the present without compromising the ability of future generations to meet their own needs" (WCED 1987). Note that a broad definition of a sustainable development also includes a sociological dimension, which will not be part of my analysis.

traditionally the economic analysis of sustainability has a strong focus on capital and its substitutability in production, the importance of incorporating preferences has been pointed out as well. Recently, Gerlagh & van der Zwaan (2002) mapped the difference between the two mentioned paradigms of sustainability into an assumption on the substitutability between the two classes of goods in the welfare function.

Another evaluative concern that is closely related to the concept of sustainability, is the demand of a precautionary handling of the uncertainty going along with the long-term environmental changes discussed in section 1.1. In this regard, Hahn & Sunstein (2005, 1) predict that "over the coming decades, the increasingly popular 'precautionary principle' is likely to have a significant impact on policies all over the world". A frequently cited definition of the *precautionary principle* is that "where an activity raises threats of harm to the environment or human health, precautionary measures should be taken even if some cause and effect relationships are not fully established scientifically" (Wingspread declaration, Raffensperger & Tickner 1999, 8). However, a major difficulty in the interpretation and application of the precautionary principle is the vagueness in its formulations. This vagueness gives rise to criticisms which are most prominently expressed by Hahn & Sunstein (2005, 1) who state that "the precautionary principle does not help individuals or nations make difficult choices in a non-arbitrary way. Taken seriously, it can be paralyzing, providing no direction at all". The authors continue that "In contrast, balancing costs against benefits can offer the foundation of a principled approach for making difficult decisions".

Let me finally address some important details of the *standard modeling framework* in environmental economics. For models under certainty, the standard approach is to depict overall welfare as a sum of per period welfare. The underlying assumption, allowing for such a welfare representation, is known as *additive separability over time*. In this model, the willingness of the decision-maker to substitute consumption between one period and another, can be characterized by the (inverse of the) *elasticity of intertemporal substitution*. An additional assumption, that is commonly adopted, is that the functions describing the welfare derived from different outcomes coincide up to a discount rate between different periods. For decision makers with an infinite planning horizon, Koopmans (1960) has shown that this structure is implied by a *stationarity* axiom, which assumes that the mere passage of time does not change preferences. Such an axiomatization depends crucially on the strict positivity of the discount rate.

For models dealing with uncertainty, the standard framework is the *expected utility* model. Here, a decision maker weights the different possible outcomes with a probability distribution, and evaluates a scenario that gives rise to a particular probability distribution by its expected welfare. For an atemporal setting, such an approach to evaluate uncertain choices has been axiomatized most prominently by von Neumann & Morgenstern (1944). In this framework, the decision maker's willingness to smooth consumption over different risk-states characterizes his degree of *risk aversion*. In models where time and uncertainty is represented simultaneously, the standard is to combine these two frameworks described above by assuming the structure of an *intertemporally additive expected utility model*. In this framework, evaluation of certain consumption streams is obtained, as before, by taking a discounted sum of per period welfare. If uncertainty prevails, the different possible outcome paths are weighted with probabilities. Then a scenario, giving rise to a particular distribution over possible consumption paths, is evaluated by its expected welfare.

A *limitation* of the latter model is that two characteristics of welfare, which are in principle independent of each other, are confined to a one-to-one relationship. Precisely, the (Arrow-Pratt) coefficient of relative risk aversion has to be *equal* to the inverse of the elasticity of intertemporal substitution. This implies that a decision maker's willingness to substitute consumption between different periods, has to coincide with his propensity to 'substitute consumption between different risk states'. Note that the definition of intertemporal substitutability and risk aversion is straightforward only in the one commodity setting. Kreps & Porteus (1978) developed a more general model, extending von Neumann & Morgenstern's (1944) axiomatic approach to a multiperiod setting (not assuming additive separability over time). Within this generalized framework, Epstein & Zin (1989) show for the one commodity case that intertemporal substitutability and risk aversion can be disentangled. This disentanglement in the Epstein-Zin framework goes along with an *intrinsic*[7] *preference for either early or late resolution of uncertainty*. The latter implies that, in such a model, uncertainty is no longer expressed immediately over different consumption paths, but has to be stated recursively over periodic outcomes (temporal lotteries). That is, uncertainty at the beginning of a period is expressed as a joint probability distribution over the *outcomes* of that period and over the *probability* distribution prevailing at the beginning of the next period. The more standard modeling approach is to depict uncertainty directly as a joint probability distribution over the outcomes in all periods.

[7]Intrinsic means not instrumental, i.e. that the information derived from an earlier resolution of uncertainty cannot be used to improve (expected) outcomes in the future.

1.3 Key Issues

In this dissertation, I develop insights and tools concerning the evaluation of long-term trade-offs between economic activity and its environmental repercussions. My study concentrates on theoretical aspects of the evaluation functional characterizing preference and welfare. I focus in particular on the weight given to the short versus the long run, and the incorporation of uncertainty. The results render contributions to the fields of environmental economics and decision theory, as well as to the more specific research areas of sustainability and cost benefit analysis. This section gives a brief overview over the key issues addressed in my dissertation.

The *first* contribution relates the *strength of the notion of sustainability* to the substitutability in welfare between environmental service flows and produced consumption streams. I analyze how the different notions of sustainability change the *weight* that is given to consumption and services in the *long run*. To this end, I study a stylized growth scenario, where the growth rate of produced consumption streams dominates that of environmental service flows. I show that if the strength of sustainability is only translated into the substitutability between environmental service streams and produced consumption, a strong notion of sustainability will generally result in lower weights given to long-run service and consumption streams than a weak notion of sustainability.

The *second* contribution is derived for the same stylized growth model. I find that in such a multi-commodity setting the optimal *social discount rates* reflect the difference in growth, as well as the degree of substitutability between the two classes of goods. These determinants are mirrored not only in the magnitudes of the social discount rates, but also in their *time development*. I point out that some degrees of substitutability can cause social discount rates to decline over time, while others can cause social discount rates to grow, both within a time consistent framework.

The *third* contribution formalizes an important aspect of the *precautionary principle*. It concerns the willingness to undergo precautionary measures in order to prevent a threat of harm. I argue that the concern of the advocates of the precautionary principle is not captured by the standard model of risk aversion, based on an intertemporally additive expected utility framework. Taking time structure more seriously, I introduce a concept called *intertemporal risk aversion*, and show its immediate relation to the concerns of the precautionary principle. This notion of uncertainty attitude is carried by standard axioms of decision theory, including those of von Neumann & Morgenstern (1944). Moreover, in contrast to the notion of standard risk aversion, the definition of intertemporal risk aversion immediately holds for the multi-commodity setting.

The *fourth* contribution of my dissertation is to work out the general time consistent model, which falls back to additively separable welfare over time for certain consumption paths, and to the von Neumann & Morgenstern (1944) setting in the atemporal case. These are the most prominent specifications for the respective framing scenarios, and I show how a careful unification allows to *disentangle risk aversion from intertemporal substitutability*. Moreover, I show how a stationary aggregation of welfare over time can be axiomatized in a finite planning horizon, without the assumption of a positive discount rate.

The *fifth* contribution analyzes the reasons for an *intrinsic preference for the timing of uncertainty resolution* in recursive utility models. I relate such a preference to the functions characterizing risk attitude and intertemporal substitutability. Moreover, I work out that a disentanglement of a decision maker's attitude with respect to risk and his propensity to substitute consumption between different periods is possible, without assuming an intrinsic preference for an early or late resolution of uncertainty. This result implies that it is possible to *disentangle* (standard) risk aversion and intertemporal substitutability also in a model, where uncertainty is expressed *non-recursively* over consumption paths.

The *sixth* contribution relates to the choice of the *pure rate of time preference*. I offer axioms under which an intertemporally risk averse decision maker chooses the pure rate of time preference as zero. These axioms concern the decision maker's stationarity of risk attitude and his attitude with respect to the timing of uncertainty resolution. While, under these axioms, a decision maker is not free to devalue the future due to pure impatience, he gives reduced weight to welfare that is uncertain. When uncertainty is increasing over time, this fact has some resemblance to discounting. However, the more a decision-maker can know about the future, the more weight it will carry.

1.4 Methodology

Part I of this dissertation employs a *stylized* growth model, which takes as given a functional representation of welfare that is additive over continuous time. The assumption of continuous time allows to adopt differential calculus to analyze the behavior of discount rates over time. I introduce the concept of good-specific discount rates as generators of marginal utility propagation. The latter concept, taken from physics, proves useful to interpret and compare different integral representations of cost benefit functionals, corresponding to different views on discounting. Having introduced the conceptual background, I reduce the general welfare function to a parametric form depicting the difference between a preference for weak and strong sustainability in an as simple as possible form that still allows to derive the main insights. In this model, the (certain) consumption levels of the two classes of goods are represented by real numbers.

In contrast, parts II and III are *axiomatic* approaches to long-term evaluation, deriving functional representations only from underlying assumptions on preferences. Again, part II derives the simplest model that is general enough to introduce and relate the relevant concepts. Then, part III starts out by extending the concepts to the most general setting in this study. Subsequently, I work out desirable restrictions that cut back on the model's complexity. In part II, objects of choice are pairs of certain present outcomes and uncertain future outcomes. Certain outcomes are depicted as elements of a compact metric space, and uncertain outcomes are represented by (Borel) probability measures on the latter. Part III makes use of a recursive extension of this 'certain×uncertain' setting to a multiperiod framework, as developed by Kreps & Porteus (1978) (temporal lotteries). It also analyzes the simpler approach where probabilities are defined directly on consumption paths. Part II and III adopt a discrete time framework. The simpler structure is considered preferable to introduce the new concept of intertemporal risk aversion, without the rather technical aspects of continuous time analysis in recursive utility models. As in the first part, I introduce a concept borrowed from physics, a so called gauge. It describes a degree of freedom in a system, here my preference representations, that can be exploited in different ways. Instead of eliminating, i.e. normalizing, this degree of freedom right at the outset of the model, the idea is to explicitly analyze this freedom. This approach proves helpful to derive a deeper understanding of the theory.

1.5 Outline

Part I of my dissertation analyzes the relation between a preference for strong versus weak sustainability and the optimal social discount rate. Anticipating the resulting non-constancy of the social discount rate, chapter 2 briefly reviews related literature on declining discount rates. Offering an interpretation as generators of marginal utility propagation, the concept of social discount rates is extended to the multi-commodity setting. For the one commodity setting, I briefly work out the well known dependence of the social discount rate on growth and intertemporal substitutability. Chapter 3 discusses the concepts of weak and strong sustainability, and relates it to the degree of substitutability in the welfare function between man-made goods and environmental service and consumption streams. Taking a closer look at the situation where consumption of produced goods grows at a faster rate than the consumption of environmental services and amenities, I work out how the limitedness in substitutability translates into modifications of the optimal social discount rates for the respective goods. Finally, chapter 4 elaborates how the good-specific social discount rates have to be employed in the cost-benefit evaluation of a (small) project. It relates the perspective of good-specific discount rates to an evaluation with a universal discount rate. Moreover, the cost-benefit evaluation is related to an evaluation by means of (imaginary) complete future markets.

Part II of my dissertation introduces the concept of intertemporal risk aversion and relates it to an important concern of the precautionary principle. In chapter 5, I introduce the precautionary principle, discuss related literature on choice under uncertainty, and motivate my approach to formalize the willingness to undergo preventive action. Chapter 6 develops the axiomatic background for the preference representation underlying the subsequent study. A particular feature of my representation is that it allows for different choices of the representing Bernoulli utility function[8] (for the same underlying preferences). I show how different normalizations (gauges) of Bernoulli utility correspond to different representations found in the literature. Chapter 7 introduces the concept of intertemporal risk aversion, and relates it to the attempt of disentangling intertemporal risk aversion from intertemporal substitutability. I show that in a multi-commodity setting the latter coefficients depend on the particular commodity under observation, and how this corresponds to a dependence on the choice of the Bernoulli utility function (gauge). Intertemporal risk aversion, on the contrary, is shown to be gauge independent. I relate the concept of intertemporal risk aversion to precaution and the concern of preventive action, and give a reinterpretation in terms of risk aversion with respect to welfare gains and losses. Finally, I introduce quantitative measures of intertemporal risk aversion and elaborate conditions for uniqueness. To focus the discussion, the axiomatizations and representations in chapters 5 to 7 use a simplified framework featuring only two periods and stationary preferences. Moreover, a zero rate of pure time preference is assumed.

Part III of my dissertation extends and refines the model of intertemporal risk aversion. First, chapter 8 axiomatizes the general representation for an arbitrary finite time horizon and non-stationary preferences, and adapts the axiomatic characterization of intertemporal risk aversion. Starting from this general perspective, chapters 9 and 10 analyze axioms that simplify the representation. In chapter 9, I examine different sta-

[8]Bernoulli utility describes a cardinal function that, by itself, represents choice over certain, one period outcomes. In combination with uncertainty and intertemporal aggregation rules, it serves as the bases for more general evaluation.

tionarity assumptions. In my finite time horizon setup, there is no canonical way to impose stationarity. While stationarity with respect to the evaluation of certain outcomes is straightforward, an axiom yielding stationarity with respect to risk attitude is more involved. Two alternative axioms are introduced. The first yields a constancy of the functionals that evaluate uncertainty in every period. The second axiom is a more natural extension of the stationarity axiom for certain outcome paths and the intuition that the mere passage of time should not change preferences. Chapter 10 analyzes a decision maker's preference with respect to the resolution of uncertainty. I relate the latter to the functions characterizing intertemporal substitutability and risk attitude, and elaborate the reasons underlying such a timing preference. I critically discuss the motivations for a non-trivial timing preference found in the related literature. I show how indifference with respect to the timing of uncertainty resolution allows to simplify the representation. I point out that the derived model allows to disentangle standard risk aversion from intertemporal substitutability in a non-recursive intertemporal expected utility framework. Finally, I combine assumptions on timing preference with assumptions on stationarity and elaborate implications for discounting. Chapter 11 concludes by summarizing the conceptual contributions, pointing out further implications and different applications, and, finally, suggesting various extensions of the study.

Part I

Social Discounting and Limited Substitutability in Welfare

Chapter 2

Social Discounting

2.1 Introduction

In the first part of my dissertation I analyze how limited substitutability in consumption between environmental services and produced goods affects the social discount rates, and the weights given to future consumption streams. The study is intimately linked with two important aspects of sustainability. First, the weight given to future consumption is the fundamental concern linking economic modeling to the definition of a sustainable development as a "development that meets the needs of the present without compromising the ability of future generations to meet their own needs" (WCED 1987). And second, the substitutability between environmental services and produced consumption is the most important distinction between the concepts of weak and strong sustainability.

My study considers a stylized growth model where the growth rate of produced consumption dominates that of environmental service streams. I generalize the concept of social discount rates to the multi-commodity setting and elaborate how social discount rates are affected by such an uneven growth. I find that the social discount rate of the environmental good, which becomes relatively more scarce over time, gets a mark down, while the rate of the produced consumption stream receives a mark up. The more challenging findings concern the behavior of the social discount rates over time. It turns out that, if the two classes of goods are of moderately limited substitutability in consumption,[1] *both* discount rates *decline* over time. The corresponding specification

[1] By moderately limited substitutability I denote the region where welfare can be derived from consuming only one of the two classes of goods, but mixtures are generally preferred to extremes. On the other hand I denote the region where 'positive welfare' cannot be gained by consuming only one of

of welfare closely relates to the concept of weak sustainability and the result seems to substantiate a demand for falling discount rates by environmentalists. In this respect, Pezzey (2006) even defines sustainable discount rates as falling discount rates. However, a surprise comes from analyzing the situation where the elasticity of substitution between the two classes of goods is smaller than one, and substitutability is strongly limited. This is a welfare specification that is closely related to the notion of strong sustainability. I find that in such a setting *both* social discount rates *grow* over time. Moreover, this growth is strong enough to imply that the weight given to consumption and environmental services in the long run becomes smaller than under moderate limitedness in substitutability, even when the environmental service stream decreases in absolute and relative turns.

I explain that the described time behavior of the social discount rates connects to a finding by Gerlagh & van der Zwaan (2002), who analyze the value share of man-made consumption in a comparable growth scenario. Furthermore, my study closely relates to a paper by Weitzman (1994) on an 'environmental discount rate' and its criticism by Arrow et al. (1995, 140). Weitzman (1994) presents a reasoning how the consideration of environmental amenities being degraded by production or being luxury goods, can lead to reduced and falling discounting rates. However, he does not model the environmental good or its value development explicitly. In particular, the interpretation of the derived quantity as a proper discount rate is criticized by Arrow et al. (1995, 140), for its lack of an explicit conversion of the environmental benefits into equivalents of produced consumption. My study can be seen as a different approach to model the relation between the development of environmental service streams and produced consumption. Unlike Weitzman, I explicitly model preferences and both classes of goods. This allows to take up the concern of Arrow et al. (1995, 140) with respect to Weitzman's (1994) work. My model does not substantiate Weitzman's (1994) result. But opposed to Arrow et al.'s (1995, 140) comment, it shows that, also when approaching the evaluation the way requested by the authors, a similar environmental reasoning can affect the time behavior of the discount rate.

More generally my model relates to a broad field of literature that motivates and works with declining (hyperbolic) discount rates for different reasons. An excellent review on declining *social* discount rates is found in Groom, Hepburn, Koundouri & Pearce (2005, 7 et seqq.) who summarize and critically review the literature on effects that can cause optimal social discount rates to decline over time. A survey on experiments indicating that observed behavior is better described by the use of falling discount rates is found

the consumption streams as the region of strongly limited in substitutability.

in Frederick, Loewenstein & O'Donoghue (2002, 378). From a different perspective, Chichilnisky (1996) and Li & Löfgren (2000) develop models of hyperbolic discounting that are based on considerations of intergenerational justice. A problem with the use of hyperbolic discount rates is that these can lead to a continuous revision of the (formerly) optimal plan, a phenomenon called time inconsistency. One way to 'solve' this problem is to look at the planning process as a non–cooperative game against one's future selves or future generations (Phelps & Pollak 1968, Arrow 1999). Another access to this problem is set forth by Weitzman (1998), Azfar (1999) and Dasgupta & Maskin (2005) who rationalize hyperbolic discounting in the case of uncertainty. In a context closer to my own Gollier (2002) developed sufficient conditions for hyperbolic discounting in an uncertain world of economic growth. From the perspective of this literature, my study provides an explanation for time consistent evaluation with non-constant social discount rates under certainty. Let me finally remark that in 2003 hyperbolic discount rates also made their way into applied policy, when the British Green Book started to prescribe hyperbolic discount rates for the evaluation of long-term projects (HM Treasury 2003, 97 et sqq.).

Part one of my dissertation is structured as follows. Subsequent to this introduction, section 2.2 introduces good-specific social discount rates and factors, and offers an interpretation of social discount rates as generators of marginal utility development over time, a concept borrowed from physics. I briefly discuss the expression for social discount rates in the standard one commodity setting. Chapter 3 analyzes how limited substitutability in welfare between environmental service streams and produced consumption affects the social discount rates in a stylized growth scenario. First, chapter 3.1 introduces the reader to the concepts of weak and strong sustainability. I relate these two concepts to different substitutability assumptions in welfare. Then, chapter 3.2 derives a reduced form for welfare that parametrizes substitutability in a simple manner and eliminates the effect of uniform growth on the social discount rates, which has been discussed extensively in the literature. Chapter 3.3 derives the results concerning magnitude and time behavior of social discount rates and the weight on future consumption streams that have been mentioned earlier in this introduction. Chapter 4 explains the cost benefit evaluation of a small project from different perspectives on discounting. In chapter 4.1 I work out how good-specific discount rates have to be treated in the process of evaluation and how the choice of numeraire affects the discount rate. The critique of Arrow et al. (1995, 140) on Weitzman's (1994) 'environmental discount rate' is taken up and reviewed. Chapter 4.2 relates the social cost benefit analysis to a market based evaluation. Finally, chapter 4.3 gives a brief summary of the results and the discussion in part I.

2.2 Social Discount Rates and Factors

This section derives social discount rates from the development of marginal utility over time. It proves beneficial for later sections to do this by simultaneously introducing social discount factors. I study the case of two (aggregate classes of) consumption goods, however, the model is easily extended to $N \in \mathbb{N}$ goods. The consumption quantities of the two goods at time t are characterized by positive real numbers, denoted $x_1(t)$ and $x_2(t)$. To simplify notation, the time argument will generally be omitted. With $\mathsf{x}_i : [0, \infty) \to \mathbb{R}$ I denote the consumption path for good i from the present $t = 0$ to the infinite time horizon. x comprises the consumption paths of the two goods in vector notation. Welfare is assumed to be representable in the form

$$\mathcal{U} = \int_0^\infty U(x_1(t), x_2(t), t)\, dt\ . \tag{2.1}$$

I call the function $U(x_1(t), x_2(t), t)$ the (instantaneous) utility function and require it to be twice differentiable. For a given consumption path x I write $U^{\mathsf{x}}(t) \equiv U(x_1(t), x_2(t), t)$. The analogous notation applies to the derivatives of U. For notational convenience the x will usually be dropped. I define the *good specific social discount factor* between time t_0 and time t as the *time propagator of marginal utility* $D_i^{\mathsf{x}}(t, t_0)$ for a given consumption path x:

$$D_i^{\mathsf{x}}(t, t_0) \equiv \frac{\frac{\partial U(x_1,x_2,t)}{\partial x_i}\big|_t}{\frac{\partial U(x_1,x_2,t)}{\partial x_i}\big|_{t_0}} \tag{2.2}$$

$$\Leftrightarrow \frac{\partial U(x_1, x_2, t)}{\partial x_i}\Big|_t \equiv D_i^{\mathsf{x}}(t, t_0)\, \frac{\partial U(x_1, x_2, t)}{\partial x_i}\Big|_{t_0} \quad , \ i \in \{1, 2\}\ .$$

It captures the value development over time by relating the value of an additional unit of consumption good x_i at time t to the value of an additional unit at time t_0. Note that in a discrete time setting such discount factors $D_i^{\mathsf{x}}(t, t_0)$ are used by Malinvaud (1974, 234).[2]

[2] In his general equilibrium setting Malinvaud calls them subjective discount factors as he considers heterogeneous agents. Looking at a social welfare framework, I call the $D_i^{\mathsf{x}}(t, t_0)$ social discount factors. Further below, I show that this wording is also appropriate in the sense that, for the one commodity standard setting, the social discount factors give rise to rates that are known as social discount rates. Obviously, the discount factors $D_i^{\mathsf{x}}(t, t_0)$ are not the same as functions of time multiplied to a time-independent instantaneous utility function (representing only pure time preference). Note

To yield the discount rates that correspond to the social discount factors one can immediately work out the relation

$$\delta_i(t) = -\frac{\frac{d}{dt}D_i^{\mathsf{X}}(t,t_0)}{D_i^{\mathsf{X}}(t,t_0)} = -\frac{\frac{d}{dt}\frac{\partial U(x_1,x_2,t)}{\partial x_i}}{\frac{\partial U(x_1,x_2,t)}{\partial x_i}} = -\frac{\frac{\partial^2 U}{\partial t \partial x_i}(t) + \frac{\partial^2 U}{\partial x_i^2}(t)\dot{x}_i + \frac{\partial^2 U}{\partial x_j \partial x_i}(t)\dot{x}_j}{\frac{\partial U}{\partial x_i}(t)}$$

for $i, j \in \{1,2\}$ with $i \neq j$. However, I also want to offer a slightly more indirect derivation, giving rise to an interesting perspective on the relationship between discount rates and factors. For this purpose, I have a closer look at the infinitesimal discount factor corresponding to the infinitesimal propagator of marginal utility:

$$\begin{aligned} D_i^{\mathsf{X}}(t+dt,t) &= \frac{\frac{\partial U}{\partial x_i}(t+dt)}{\frac{\partial U}{\partial x_i}(t)} = 1 + \frac{\frac{\partial U}{\partial x_i}(t+dt) - \frac{\partial U}{\partial x_i}(t)}{\frac{\partial U}{\partial x_i}(t)} \\ &= 1 + \frac{\frac{\partial^2 U}{\partial t \partial x_i}(t) + \frac{\partial^2 U}{\partial x_i^2}(t)\dot{x}_i + \frac{\partial^2 U}{\partial x_j \partial x_i}(t)\dot{x}_j}{\frac{\partial U}{\partial x_i}(t)} dt \\ &\equiv 1 - \delta_i(x(t),\dot{x}(t),t)\,dt \equiv 1 - \delta_i(t)\,dt \end{aligned}$$

for $i, j \in \{1,2\}$ with $i \neq j$. The instantaneous change of D_i^{X} is completely characterized by the term $\delta_i(t) \equiv \delta_i(x(t),\dot{x}(t),t)$, which is the discount rate corresponding to D_i^{X}. In physics (the negative of) $\delta_i(t)$ is called the generator of D_i^{X} as it describes – or from an active point of view generates – the change of D_i^{X}.[3] In the context of this study, $\delta_i(t)$ can be understood as the (good-specific) generator of time development of marginal utility. Therewith, $\delta_i(t)$ is the generator of value development for an additional unit of good x_i in the future.[4] The finite time propagator follows from the infinitesimal one as derived in Appendix A.1 yielding:

$$\begin{aligned} D_i^{\mathsf{X}}(t,t_0) &= \exp\left(-\int_{t_0}^{t} \delta_i(x(t'),\dot{x}(t'),t')\,dt'\right) \qquad (2.3) \\ &= \exp\left(\int_{t_0}^{t} \frac{\frac{\partial^2 U}{\partial t' \partial x_i}(t') + \frac{\partial^2 U}{\partial x_i^2}(t')\dot{x}_i + \frac{\partial^2 U}{\partial x_j \partial x_i}(t')\dot{x}_j}{\frac{\partial U}{\partial x_i}(t')}\,dt'\right) . \end{aligned}$$

that the $D_i^{\mathsf{X}}(t,t_0)$ can also be calculated if the pure time dependence of instantaneous utility is not multiplicatively separable.

[3] Compare Sakurai (1985, 46 et sqq.,71 et sq.) or Goldstein (1980, chapter 9) for this view on classical and quantum mechanics (e.g. momentum being the generator of translation). The minus sign is introduced to meet the economic perspective of positively discounting instead of negatively "upcounting".

[4] Note that from a technical perspective the characterization of $\delta_i(t)$ as the generator of marginal utility propagation is much less ambiguous than simply calling it "social" or "subjective" discount factor (compare footnote 2).

In models with a single (aggregate) consumption good, $\delta_i(t)$ is known as the (instantaneous) social rate of time preference or *social discount rate*. This stands out more clearly if instantaneous utility is specified to the form usually applied in discount utility models: $U(x_1, x_2, t) = u(x_1, x_2)e^{-\rho t}$. For the moment, let me neglect the second commodity by setting it constant.[5] Then, the discount rate $\delta \equiv \delta_1$ becomes

$$\delta(t) = \rho - \frac{\frac{\partial^2 u}{\partial x_1^2}}{\frac{\partial u}{\partial x_1}} \dot{x}_1 = \rho - \frac{\partial \frac{\partial u}{\partial x_1}}{\partial x_1} \frac{x_1}{\frac{\partial u}{\partial x_1}} \frac{\dot{x}_1}{x_1} = \rho + \theta_{(x(t))}\, \hat{x}_{1(x_1(t),\dot{x}_1(t))} \; . \tag{2.4}$$

This expression for the social discount rate is well known in the literature, see for example Arrow et al. (1995, 136) or Groom et al. (2005). The constant ρ is called the pure rate of time preference. The term θ depicts the (absolute[6] of the) elasticity of marginal utility of consumption, which is the inverse of the intertemporal elasticity of substitution. Finally $\hat{x}_1$ denotes the growth rate of the consumption commodity. Equation (2.4) states that the value development of an additional unit of good x_i is generated by the pure rate of time preference as well as a term proportional to the growth rate of consumption and the elasticity of marginal utility. To work out the intuition of the second term, assume that consumption is growing over time. Then, an individual with a decreasing marginal valuation of consumption will evaluate an additional unit of consumption in the future less than in the present. Therefore, he discounts consumption at a higher rate than just pure time preference. This effect is proportional to the growth rate and the measure θ for the relative decrease in marginal valuation as overall consumption increases. In most macroeconomic models the function u is assumed to exhibit constant elasticity of intertemporal substitution (CIES). This assumption implies that in a steady state the term $\theta\, \hat{x}_1$ and, thus, the social discount rate $\bar{\delta} = \rho + \theta \hat{x}_1$ is constant. A constant rate of discount goes along with exponential discounting of future consumption, i.e. the discount factor in equation (2.3) becomes $D^{\mathsf{x}}(t, t_0) = e^{-\bar{\delta} t}$.

In general, the terms in equation (2.4) must not necessarily be constant. In fact, the term $\theta\, \hat{x}_1$ is also used to argue for hyperbolic discounting. A discount function is said to be *hyperbolic* if it is characterized by a falling instantaneous discount rate (Laibson 1997, 450). Dasgupta (2001, 183 et sqq.) works out that in the face of global climate change, a decline in consumption growth $\hat{x}_1$ would imply that social discount rates should fall over time. It is immediate from equation (2.4) that this effect is proportional to the absolute of the elasticity of marginal utility θ. In other words, a *lower intertemporal elasticity of*

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[5] That is, x_2 can be regarded as a fixed parameter of the utility function.

[6] As in the standard DU models diminishing but positive marginal utility in consumption is assumed, the term $-\frac{\partial^2 u}{\partial x_1^2} x_1 / \frac{\partial u}{\partial x_1} = \theta$ turns out to be positive.

substitution (θ^{-1}) intensifies the effect that a *decline in growth* induces decreasing social discount rates and thus a relatively *higher weight given to the future*. I will come back to this point at the end of the next chapter. Finally, Gollier (2002) works out conditions under which a translation of equation (2.4) into a framework with uncertainty can lead to a falling discount rate by the term $\theta\, \hat{x}_1$. In the next chapter, I will analyze how the explicit modeling of two commodity classes, which are limited in substitutability, changes equation (2.4) and the weight given to future consumption. Such a weighing of future consumption is an important concern of a sustainable development. Moreover, when identifying the commodity classes x_1 and x_2 with environmental service streams and produced consumption, the limitedness in substitution directly relates to the two opposing concepts of weak and strong sustainability.

Chapter 3

Sustainability and Limited Substitutability

3.1 A Preference for Weak versus Strong Sustainability

Concerning the environment-economy interaction there are two different paradigms of sustainability. Both of them are carried by the broad definition of a sustainable development as formulated in the report of the World Commission on Environment and Development (Brundtland report) which defines that "sustainable development is development that meets the needs of the present without compromising the ability of future generations to meet their own needs" (WCED 1987). The paradigm of *weak sustainability* translates this requirement into the demand that overall welfare should not decline over time. To this end, advocates of the weak sustainability paradigm allow for a substitution between man-made and environmental capital. The paradigm of *strong sustainability* requires that the value (or the physical amount) of natural capital (or its service flows) should be non-declining itself. The reason for the latter demand is that the advocates of the strong sustainability paradigm do not believe in substitutability between the different types of capital. Therefore, they translate the requirement of not compromising future generations into the demand to individually maintain natural, economic and social capital. While these three types of capital correspond to the different dimensions of sustainability, natural capital is frequently broken down further into different classes of capital corresponding to different service flows. Then, each of these classes is required to be non-declining by itself (either in physical terms or in value). For an overview over the more detailed differences between weak and strong sustainability as well as further differentiations of sustainability demands compare for example Neumayer (1999) and van den Bergh & Hofkes (1998).

Traditionally the economic analysis of sustainability is mostly focused on capital and its substitutability *in production*. However, as soon as one acknowledges that a perfect replication of natural capital and its service flows by man-made capital is not possible, the evaluative aspect of weighing natural service flows against man-made service flows gains in importance. Examples for non-substitutable environmental assets pointed out by Pearce, Markandya & Barbier (1997, 37) include the ozone layer and its UV-protection function, the climate-regulating functions of ocean phytoplankton, the nutrient trap and pollution cleaning functions of wetlands and the watershed protection functions of tropical rain-forests. Given such a non-substitutability, one either has to adhere to an extreme notion of strong sustainability and leave the mentioned assets untouched, or one has to allow for a trade-off and make the respective service flows comparable to other service flows. For this purpose the substitutability of the different service flows *in welfare* has to be specified. Note, that few people would claim that, for example, the ozone layer as a whole can be replaced by man-made goods or services. However, it can be argued for a substitutability at the (current) margin. For example, sunscreen lotion or shelter under glass can protect from ultraviolet radiation and, thus, 'replace' a little of the stratospheric ozone. But, observe that such an argument already involves the welfare judgment that taking a sun bath with or without sunscreen are complete substitutes in consumption, or that a glass roof is a substitute to the open air. While some might agree to such a statement, others will usually contradict. In particular, this disagreement over the substitutability between natural consumption and service streams and its potential technical substitutes represents a difference in the preferences between the advocates of the strong and the advocates of the weak sustainability concept.

The importance of explicitly incorporating preferences concerning environmental service flows and environmental quality into the analysis of a sustainable development is, for example, pointed out by Pearce et al. (1997, 33). From a fairly comprehensive point of view van den Bergh & Hofkes (1998, 14) summarize the motivation for the standpoint of strong sustainability as "the recognition that natural resources are essential inputs in economic *production, consumption or welfare* that cannot be substituted for by physical or human capital, or the acknowledgement of environmental integrity rights in nature". Note that the latter 'environmental integrity rights' are often referred to as intrinsic values of nature. These values are generally regarded to be non-anthropocentric concepts, based on moral, ethical or religious considerations.[1] I will absorb them to the degree possible into the neoclassical preference framework as a generalized (existence) value of natural capital and its service flows. Explicitly, my model only considers service and consumption *flows* (compare equation 2.1). An existence value attributed to a particular capital at a certain point of time has to be captured as an 'existence service flow' proportional to the amount of existing capital. Moreover, the words service flow, consumption good and amenity stream will be used interchangeably.

Taking a non-perfect technical substitutability between natural and man-made capital as given, my analysis focuses on the limitedness in substitution between the corresponding service flows in social welfare. Social welfare will be depicted by a constant elasticity of substitution (CES) function which allows a straight forward analysis of different degrees of substitutability between the two aggregate classes of produced goods and environmental services. I call the region in which welfare can be derived from consuming only either of the two commodity classes, the region of *moderate limitedness in substitutability* and identify it with a *weak sustainability preference.* With these preferences, a mixture between the two classes of consumption and service flows is generally preferred. However, it is possible that a decision maker agrees to a trade-off that deprives him completely of his natural capital and the according service flows. I refer to the region in which it is not possible to derive welfare only from produced consumption streams, as the region of *strongly limited substitutability* and identify it with the notion of strong sustainability, calling it a *strong sustainability preference.*[2] The extreme case of a strong sustainability preference would correspond to a preference relation that does not allow for any trade-off between the two classes of consumption. The extreme of weak sustainability preference would limit to perfect substitutability in consumption between produced and natural service flows. In between these two extremes there is a wide range of parameters that correspond to more moderate standpoints.

A comparable translation of the two notions of sustainability into a preference or welfare framework has been set out by Gerlagh & van der Zwaan (2002). Their analysis focuses on the (infinitely) long run in a growth scenario where produced consumption

[1] I will not delve into the philosophical discussion about the meaning of a non-anthropocentric value system. For a further discussion of a potential foundation and the difficulties going along with the concept of intrinsic value and the construction of non-anthropocentric environmental ethics compare for (different perspectives) Buchdahl & Raper (1998), Grey (1993) and most critical to the meaningfulness of a non-anthropocentric ethics Lynch & Wells (1998).

[2] In the language of Dasgupta & Heal (1974, 4) for substitutability in production, such a functional specification corresponds to, here a preference, where both inputs are *essential.*

grows to infinity, while the flow of environmental goods is bounded. Against this background the authors distinguish between poor and perfect long-run substitutability by a characteristic of the welfare function for an infinitely grown produced consumption stream. It turns out that in the CES welfare characterization used in my setup, Gerlagh & van der Zwaan's (2002) distinction between poor and perfect long-run substitutability and my distinction between the regions of strongly and of moderately limited substitutability coincide (Gerlagh & van der Zwaan 2002, 335). Therefore, Gerlagh & van der Zwaan's (2002) mapping of the notions of weak versus strong sustainability into the welfare specification matches my notions of weak and strong sustainability preference. The next section analyzes how these notions of weak versus strong sustainability preference are reflected in the social discount rates, in a scenario where produced consumption grows at a faster rate than the consumption of environmental service streams. Particular attention is paid to the question of weighing the long versus the short run.

3.2 Limited Substitutability in Consumption and Social Discount Rates

Returning to the model set out in equations (2.1-2.3) the 'commodity' x_1 is now interpreted as a flow of environmental goods and services while x_2 is representing an aggregate of produced consumption. Moreover, I assume an exponential time dependence of instantaneous utility yielding $U(x_1, x_2, t) = u(x_1, x_2)e^{-\rho t}$. This functional form corresponds to a constant rate of pure time preference ρ and implies time consistency of the planning functional (2.1). In such a multicommodity setting equation (2.4) generalizes, and the discount rate corresponding to the social discount factor $D_1^{\mathbf{x}}(t, t_0)$ becomes

$$\delta_1(t) = \rho - \frac{\frac{\partial^2 u}{\partial x_1^2}}{\frac{\partial u}{\partial x_1}}\dot{x}_1 - \frac{\frac{\partial^2 u}{\partial x_1 \partial x_2}}{\frac{\partial u}{\partial x_1}}\dot{x}_2 \ . \tag{3.1}$$

It comprises an additional term that depends on the substitutability[3] $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ between the two classes of goods.[4] To work out the influence of substitutability in welfare on the social discount rate and its evolvement over time, I take instantaneous utility to be of the functional form $u(x_1, x_2) = [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{1/s}$ with $s \in \mathbb{R}, a_1, a_2 \in$

[3] See Coto-Millán (1999, 21) for different ways of defining substitutability of consumption goods. Hereafter I will go over to a functional form that directly parametrizes substitutability.

[4] Note that equation (3.1) has independently been derived by Weikard & Zhu (2005) who also comment on the magnitude effects (see below) but do not analyze time behavior of the discount rates.

$\mathbb{R}_{++}, a_1 + a_2 = 1$ and $u_1, u_2 \geq 0$.[5] This step furthers understanding as it separates good-specific utility $u_i(x_i)$ from substitutability effects parameterized in a simple form by s. As derived in appendix A.1, for such a welfare specification the social discount rate for the environmental service stream becomes

$$\delta_1(t) = \rho - \frac{\frac{\partial^2 u_1}{\partial x_1^2}}{\frac{\partial u_1}{\partial x_1}} \dot{x}_1 - (1-s) \frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \left(\frac{\frac{\partial u_2}{\partial x_2}(x_2)}{u_2(x_2)} \dot{x}_2 - \frac{\frac{\partial u_1}{\partial x_1}(x_1)}{u_1(x_1)} \dot{x}_1 \right) . \quad (3.2)$$

The first and the second term in equation (3.2) resemble the widely used equation (2.4). In the following I want to examine the additional third term that depends on the substitutability parameter s. With this objective in mind, I simplify the utility function by setting $u_1(x_1) = x_1$ and $u_2(x_2) = x_2$, which leads to the standard CES utility function $u(x_1, x_2) = [a_1 x_1^s + a_2 x_2^s]^{1/s}$.[6] Thereby, I eliminate in equation (3.2) the well studied second term and simplify the third without changing its dependence on the substitutability parameter s. This step leads to the discount rate:

$$\delta_1(t) = \rho - (1-s) \underbrace{\frac{a_2 x_2^s}{a_1 x_1^s + a_2 x_2^s}}_{\equiv \mathit{Vsp}^s(x_1, x_2)} (\hat{x}_2 - \hat{x}_1) . \quad (3.3)$$

The first determinant in the social discount rate for the environmental service stream in equation (3.3) is the pure rate of time preference ρ. It is reduced by a second term which comprises three different components. The first component $(1 - s)$ *is a measure for the limitedness in substitutability* between the two classes of goods. The second component depicts the *v*alue *s*hare of the *p*roduced consumption stream:

$$\mathit{Vsp}^s(x_1, x_2) = \frac{\frac{\partial u}{\partial x_2} x_2}{\frac{\partial u}{\partial x_1} x_1 + \frac{\partial u}{\partial x_2} x_2} = \frac{a_2 x_2^s}{a_1 x_1^s + a_2 x_2^s} . \quad (3.4)$$

It depends on the ratio $\frac{x_1}{x_2}$ between the environmental services and the produced goods consumed,[7] the utility weights a_1 and a_2 and the substitutability parameter s. The last component in equation (3.3) characterizes the relative growth overweight of the produced

[5] $\mathbb{R}_{++}$ denotes the strictly positive real numbers. For $s = 0$ the function is defined by the limit $s \to 0$ yielding $u(x_1, x_2) = u_1(x_1)^{a_1} u_2(x_2)^{a_2}$. For $s \to -\infty, \infty$ the limit functions are $\min\{u_1(x_1), u_2(x_2)\}$ and $\max\{u_1(x_1), u_2(x_2)\}$ respectively. $u_i \geq 0$ stands for $u_i(x_i) \geq 0$ for all $x_i \in \mathbb{R}_+$.

[6] CES functions exhibit a constant elasticity of substitution σ that relates to s by the formula $\sigma = \frac{1}{1-s}$. For its derivation see Arrow, Chenery, Minhas & Solow (1961). Observe that CES functions are homogeneous of degree one. Thus, proportional overall growth does not change marginal utility as it follows that the latter is homogeneous of degree zero in consumption. This explains why the chosen functional form is so well suited to focus on the new effect due to relative difference in growth, filtering out the overall growth effect extensively discussed in the literature in connection with equation (2.4).

[7] This is easily observed by multiplying the nominator and denominator on the right hand side of equation (3.4) with x_2^{-s}.

consumption stream with respect to the environmental service stream. Altogether the second term on the right hand side of equation (3.3) can be summarized as follows. The relative growth overweight in produced consumption is weighted with the value share of produced consumption. This expression is then weighted with the limitedness in substitutability between produced and environmental amenity streams and subtracted from the pure rate of time preference. A closer analysis of the expression for different degrees of substitutability will be the object of investigation of the next section.

Before, however, let me derive the analogous social discount rate for produced consumption and service streams. It is easily arrived at by switching the indices in equation (3.3):

$$\delta_2(t) = \rho - (1-s) \underbrace{\frac{a_1 x_1^s}{a_2 x_2^s + a_1 x_1^s}}_{\equiv \mathit{Vse}^s(x_1,x_2)} (\hat{x}_1 - \hat{x}_2) \ . \tag{3.5}$$

The interpretation is analogous to that of equation (3.3). This time the growth overweight of the environmental service stream is weighted with the *v*alue *s*hare of the *e*nvironmental services

$$\mathit{Vse}^s(x_1,x_2) = \frac{\frac{\partial u_1}{\partial x_1} x_1}{\frac{\partial u_1}{\partial x_1} x_1 + \frac{\partial u_2}{\partial x_2} x_2} = \frac{a_1 x_1^s}{a_1 x_1^s + a_2 x_2^s} \ .$$

The resulting expression is again weighted with the limitedness in substitutability between produced and environmental consumption and then subtracted from the pure rate of time preference. In the next section I analyze a scenario where the environmental amenity and service streams grow at a slower rate than produced consumption. Anticipating such a growth overweight on the part of produced consumption, let me rearrange equation (3.5) to the form

$$\delta_2(t) = \rho + (1-s)\ \mathit{Vse}^s(x_1,x_2)\ (\hat{x}_2 - \hat{x}_1) \ . \tag{3.6}$$

3.3 Social Discount Rates in a Stylized Growth Scenario

In the following, I want to analyze how the weights for future consumption evolve in a scenario where produced consumption grows at a faster rate than consumption of environmental services. The underlying assumption is that technological progress increases the availability of produced consumption at a faster rate than the availability of environmental service and amenity streams can be increased. In fact, when thinking about the essential life-support services that most advocates of a notion of strong sustainability are concerned about, like for example climate regulation functions (compare section 3.1), it is hard to think about a long-term positive growth rate of environmental services at all. When thinking about simpler environmental consumption goods like defined in Fisher & Krutilla (1975, 360) as goods that are "generally consumed on site, with little or no transformation by ordinary productive processes", like for example scenic views, then by definition these goods are not or little affected be technological progress in production.[8] The appreciation of biodiversity and its existence value is another example where the growth rate of the corresponding existence service flow is negative and a serious growth within a human planning horizon is hard to imagine. Against this background I introduce the following

Assumption 1: The growth rate of produced consumption and services is higher than the growth rate of environmental amenity streams and services, that is $\hat{x}_2(t) > \hat{x}_1(t)\ \forall\, t$.

The assumption allows of course for a decline in environmental goods and services. It also allows for a scenario, which is sometimes put forth in relation to climate change, where production and environment decline together, as long as environmental service flows decline at a higher rate.

Under this stylized growth assumption, I want to analyze how different specifications of the limitedness in substitutability in welfare, between the two classes of goods and services, affects the weights given to future consumption. Hereby I want to focus on the effect that stems from the interplay of the difference in growth rates and the limitedness in substitutability. As I have worked out in section 3.2, a simple welfare function serving this purpose is the following.

Assumption 2: Welfare is representable in the functional form
$\mathcal{U} = \int_0^\infty [a_1 x_1^s + a_2 x_2^s]^{1/s} e^{-\rho t}\, dt$ with $a_1, a_2 \in \mathrm{I\!R}_{++}, a_1 + a_2 = 1$ and $s \in \mathrm{I\!R}, s \leq 1$.[9]

As elaborated in the preceding section this specification of welfare disregards the influence of an even overall growth (or decline) on the social discount rates and focuses on the substitutability effect. I added the assumption that the substitutability parameter s should be smaller or equal to one. The range $s \in (1, \infty)$ corresponds to the

[8] One can think of several cases, where technological progress helps accessing or enjoying environmental goods. However, such a complementarity between produced and environmental goods and services is captured in the welfare function, i.e. in the parameter s.

[9] For $s = 0$ the integrand is defined by limit, yielding the well known Cobb-Douglas specification: $\lim_{s\to 0}[a_1 x_1^s + a_2 x_2^s]^{1/s} = x_1^{a_1} x_2^{a_2}$ (Arrow et al. 1961, 231).

assumption that environmental and produced services are 'more than perfect substitutes' in the sense that extreme choices tending towards the consumption of only one of the two consumption streams are generally preferred to mixtures. Such an assumption would neither seem reasonable when analyzing environmental and produced consumption and service streams, nor would it correspond to any notion of sustainability. In the following, I analyze one after another the scenarios corresponding to $s = 1$ (perfect substitutability), $s = 0$ (Cobb-Douglas preferences), $s \in (0, 1)$ (moderate limitedness in substitutability) and $s < 0$ (strong limitedness in substitutability).

The interpretation of the social discount rates derived for these different welfare specifications is the following. I take as given an underlying growth scenario that satisfies assumption 1. A decision-maker or social planner is asked to evaluate a small project that affects environmental service streams and produced consumption streams over some period of time. Then, the social discount rates and factors specify the weight that a planner, subscribing to a particular welfare specification, gives to the corresponding future consumption streams. In particular, I will be interested in the time development of these weights and the differentiation between the scenarios of moderate and strong limitedness in substitutability that I identified with the notions of weak versus strong sustainability in section 3.1. A formal setup of such a small project evaluation is given in chapter 4.[10]

The case of **perfect substitutability** in consumption between environmental service flows and produced consumption corresponds to the substitutability parameter $s = 1$. It implies additivity in welfare between the different classes of goods $u(x_1, x_2) = a_1x_1+a_2x_2$ and an elasticity of substitution $\sigma = \frac{1}{1-s}$ that is infinite. As there is no limitedness in substitutability ($1-s = 0$), equations (3.3) and (3.6) show that the social discount rates for both classes of goods coincide with the pure rate of time preference: $\delta_1 = \delta_2 = \rho$. Note that this result holds by construction (and reduction) of the welfare function carried out in section 3.2 to focus on the substitutability effect and disregard other growth effects.

The next preference specification that I want to analyze corresponds to the widely used **Cobb-Douglas** welfare function $u(x_1, x_2) = x_1^{a_1} x_2^{a_2}$ which is implied by $s = 0$ (Arrow et al. 1961, 231). This welfare specification lies at the border between the regions identified with the notion of weak and strong sustainability. For such a welfare specification the elasticity of substitution $\sigma = \frac{1}{1-s}$ equals unity. The value share of the environmental service stream $Vse^{s=0}$ corresponds to its utility weight a_1 and the value

[10]Smallness of the project means that the changes brought about by the project do not affect the overall growth scenario.

share of the produced consumption stream corresponds to $Vsp^{s=0} = a_2$. Therefore, the social discount rates corresponding to equations (3.3) and (3.6) become

$$\delta_1(t) = \rho - a_2\,(\hat{x}_2 - \hat{x}_1) \quad \text{and} \tag{3.7}$$

$$\delta_2(t) = \rho + a_1\,(\hat{x}_2 - \hat{x}_1)\ . \tag{3.8}$$

The social discount rate for the environmental service flow is *reduced* by a term proportional to the difference in the growth rates and to the (relative[11]) utility-weight given to produced consumption. In a *steady state* the terms in equation (3.7) are constant and therefore *discounting of the environmental good stays exponential with a lower discount rate.* The *social discount rate for produced consumption is also constant* in a *steady state.* However, it is *augmented* by a term proportional to the difference in growth rate and to the (relative) utility-weight given to environmental amenities and services. These results are summarized in

Proposition 1: Let assumptions 1 and 2 hold with $s = 0$.

Then the social discount rates are given by equations (3.7) and (3.8). The *social discount rate for the environmental service stream receives a mark down* proportional to the difference in growth rates and the utility weight given to the produced consumption stream. The *social discount rate for the produced consumption stream receives a mark up* proportional to the difference in growth rates and the utility weight given to the environmental consumption stream. *In a steady state both social discount rates are constant.*

The intuition is straight forward. The slower growing environmental consumption good becomes relatively more scarce as time evolves. Therefore, the social discount rate, expressing its value development over time, gets a mark down corresponding to a higher weight given to future environmental service streams. On the other hand, the produced good becomes more abundant in relative terms and, therefore, its social discount rate receives a mark up. The fact that the mark up/down of the goods is proportional to the value share corresponding to the utility weight given to the *other* consumption good, is best understood by considering the situation where the other good receives a (close to) zero utility weight. Then the evaluation of the first good should not (or very little) depend on the evolvement of the second consumption stream. Therefore, as the utility weight given to a good goes to zero, it should be the social discount rate of the other good that is not affected anymore by limitedness in substitutability.

After these two rather specific parameter constellations where $s \in \{0, 1\}$, I turn to

[11]Note that $a_2 = \frac{a_2}{a_1 + a_2}$ as $a_1 + a_2 = 1$.

the range where the substitutability parameter lies anywhere in between, i.e. $0 < s < 1$. This case goes along with substitution elasticities between one and infinity. As motivated in section 3.1 I call this parameter range a region of **moderate limitedness in substitutability** because utility can be gained by consuming only one class of service streams, but mixtures are preferred. Connecting to the sustainability debate, I have identified these preferences with the notion of **weak sustainability**, as - with such a welfare specification - there is no a priory limit for substituting nature by an increase in production. Analyzing the social discount rate for the environmental good, equation (3.3) shows that again a positive term is deducted from the pure rate of time preference. However, as I show in the proof of proposition 2, the term Vsp^s in the mark down is monotonously increasing over time. On the other hand, the social discount rate for the produced consumption stream still gets a mark up (equation 3.6). However, the term Vse^s in the mark up is strictly decreasing over time.

Proposition 2: Let assumptions 1 and 2 hold with $s \in (0, 1)$. Then the social discount rates are given by equations (3.3) and (3.6).
The *social discount rate for the environmental service stream receives a mark down* proportional to the difference in growth rates, the value share of the produced consumption stream and the limitedness in substitutability expressed by $(1 - s)$.
The *social discount rate for the produced consumption stream receives a mark up* proportional to the difference in growth rates, the value share given to the environmental consumption stream and the limitedness in substitutability expressed by $(1 - s)$.
In a steady state, *both social discount rates fall over time* to $\lim_{t\to\infty} \delta_1 = \rho - (1 - s)(\hat{x}_2 - \hat{x}_1)$ and $\lim_{t\to\infty} \delta_2 = \rho$ and discounting is hyperbolic.

I show more generally in the proof of proposition 1 that the existence of $\epsilon > 0$ and $t^* \in [0, \infty)$ with $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all $t \geq t^*$ is enough to ensure that in the long run the discount rates approach the form $\delta_1(t) = \rho - (1 - s)\left(\hat{x}_2(t) - \hat{x}_1(t)\right)$ and $\delta_2(t) = \rho$. However, outside of a steady state a strong fluctuation in the difference in growth rates can bring about a time period in which either of the discount rates is constant or increasing. Observe furthermore that the discount rate for the environmental service stream x_1 will eventually *grow negative* if there exists t^* such that $(1-s)(\hat{x}_2(t)-\hat{x}_1(t)) > \rho \ \forall \ t > t^*$. That is, if the difference in the growth rates between the two classes of services, weighted with the limitedness in substitutability, dominates the rate of pure time preference ρ.[12]

[12]Observe that this relation determines only the instantaneous discount rate. In addition it is also

To interpret the result let me first compare it to the result of proposition 1. The two social discount rates still receive a comparable mark up/down which is proportional to the difference in growth rates. I first have a closer look at the social discount rate for the environmental service stream. In proposition 2, the effect on δ_1, triggered by the growth overweight of the produced consumption stream, is no longer simply proportional to the utility weight given to the produced consumption stream. It is weighted with the value share of the produced consumption stream and the limitedness in substitutability. The limitedness in substitutability happened to be one in proposition 1. Moreover, the Cobb-Douglas specification of welfare in proposition 1 has the unique feature that the value share of a commodity x_i corresponds to its utility weight a_i independent of the consumption levels. In general however, the value share Vsp^s, which is weighing the importance of the growth overweight of produced consumption for the evaluation of the environmental service stream, depends on the consumption levels. Due to the relative increase of produced consumption this value share grows over time. The higher it gets, the more attention is paid in δ_1 to an increasing relative scarcity with respect to the produced consumption stream. This increasing attention to the relative scarcity causes the social discount rate to decline and the weight given to future environmental services to increase over time (as compared to a scenario with perfect substitutability or no change in relative scarcity). Moreover, such a relative scarcity is only important to the degree that the two classes of goods are limited in substitutability. A similar reasoning holds true for the social discount rate of the produced consumption stream. Here the growth 'overweight' of the other commodity is negative resulting in a mark up of the discount rate. As the value share of the environmental good Vse^s declines, the attention payed to such a mark up in the social discount rate for the produced consumption stream falls over time. Therefore, the discount rate of the produced consumption stream is declining as well.

Finally, let me turn to the welfare evaluation that goes along with a substitutability parameter $s < 0$. This parameter range corresponds to an elasticity of substitution smaller than unity. As I have motivated in section 3.1, I call such a parameter range a region of **strong limitedness in substitutability**, because welfare cannot be gained by consuming only one of the two classes of service streams. In connection with the sustainability discussion, I identified these preferences with the notion of **strong sustainability** as they impose a limit on substituting nature by an increase in production. Analyzing the social discount rate for the environmental service stream, equation (3.3) shows that, like in the other examined scenarios, a positive term is deducted from the

possible that the social discount factor $D_i^{\mathbf{x}}(t, t_0)$ grows bigger than 1.

pure rate of time preference. However, as I show in the proof of proposition 3, the term Vsp^s is now declining over time. On the other hand, the social discount rate for the produced consumption stream still gets a mark up (equation 3.6). However, the term Vse^s in the mark up now is strictly increasing over time.

Proposition 3: Let assumptions 1 and 2 hold with $s < 0$. Then the social discount rates are given by equations (3.3) and (3.6).
The *social discount rate for the environmental service stream receives a mark down* proportional to the difference in growth rates, the value share of the produced consumption stream and the limitedness in substitutability expressed by $(1 - s)$.
The *social discount rate for the produced consumption stream receives a mark up* proportional to the difference in growth rates, the value share given to the environmental consumption stream and the limitedness in substitutability expressed by $(1 - s)$.
In a steady state, *both social discount rates grow over time* to $\lim_{t\to\infty} \delta_1 = \rho$ and $\lim_{t\to\infty} \delta_2 = \rho + (1 - s)(\hat{x}_2 - \hat{x}_1)$.

The first part of the proposition is equivalent to that of proposition 2. The interesting difference is, however, the evolvement of the optimal social discount rates over time. Recall that the the two scenarios analyzed in propositions 2 and 3 only differ in the assumption about the substitutability between environmental service streams and man-made service and consumption streams. Proposition 2 considers the case where there is moderate limitedness in substitutability, corresponding to a notion of weak sustainability, and yields that optimal social discount rates should be falling over time. Proposition 3 considers the case where there is strong limitedness in substitutability, corresponding to the notion of strong sustainability, and yields that optimal social discount rates should be growing over time. This result is rather surprising as the usual intuition as expressed for example in Groom et al.'s (2005, 2) survey on declining discount rates is that "It is immediately obvious that using a declining discount rate would make an important contribution towards meeting the goal of sustainable development" and Pezzey (2006) even defines sustainable discount rates as falling discount rates. Now the preferences underlying proposition 3 relate to a stronger concept of sustainability than those of proposition 2. Nevertheless, it is proposition 3 that brings about optimal social discount rates that increase over time. Even more surprisingly might be the following result which is an immediate consequence of the steady state results of propositions 2 and 3.

Proposition 4: Evaluating the social discount rates for a *given growth scenario* under the assumptions 1 and 2 the following assertion holds.

In a steady state, the *long-term social discount rates corresponding to a strong sustainability preference* ($s < 0$) *are higher than those implied by a weak sustainability preference* ($0 < s < 1$). That is, $\lim_{t\to\infty} \delta_i^{s<0}(t) > \lim_{t\to\infty} \delta_i^{0<s<1}(t)$ for $i \in \{1, 2\}$. For a similar statement without a steady state, assume the existence of $\epsilon > 0$ and $t^* \in [0, \infty)$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all $t \geq t^*$. Then, there exists $\bar{t} \in [0, \infty)$ such that $\delta_i^{s<0}(t) > \delta_i^{0<s<1}(t)$ for all $t > \bar{t}$ and $i \in \{1, 2\}$.

A numerical example for the time evolvement of the social discount rates for the two different scenarios is drawn in figure 3.1. In the left diagram the substitutability parameter is chosen to be $s = .5$ corresponding to moderate limitedness in substitutability and a weak sustainability preference. In the right diagram the substitutability parameter is chosen to be $s = -.5$ corresponding to strong limitedness in substitutability and a strong sustainability preference. The other parameters are chosen equally for both scenarios as $\rho = 3\%, \hat{x}_2 - \hat{x}_1 = 2.5\%$ and $a_1 = a_2 = .5$.[13] As the model is constructed to only depend on the relative growth difference, this scenario depicts equally well a situation where both growth rates of consumption are positive (e.g. $\hat{x}_2 = 3\%$ and $\hat{x}_1 = .5\%$), a scenario where produced consumption grows and environmental services decline (e.g. $\hat{x}_2 = 1.5\%$ and $\hat{x}_2 = -1\%$), or one where both forms of cosumptions are subject to a decrease over time. The mark up/down as well as the time behavior pointed out in propositions 2 and 3 is clearly observable. Moreover, after $t = 88$ years, the (instantaneous) discount rate for the environmental service stream grows bigger for the strong sustainability scenario than for the weak sustainability scenario. Note that this does not immediately imply that the weight given to the environmental service stream is lower with a strong sustainability preference. As derived in chapter 2.2, the evaluation of an extra unit of environmental services is captured by the corresponding discount factor (the propagator of marginal utility). For the same scenario specifications as in figure 3.1 I have depicted the corresponding discount factors in figure 3.2. This picture seems to go along better with the intuition that under a stronger sustainability preference future environmental service streams should receive a higher weight. The reason is found in equation (2.3). A discount factor relates to the rate by $D_i^{\mathbf{x}}(t, t_0) = e^{-\int_{t_0}^{t} \delta_i(x(t'), \dot{x}(t'), t') dt'}$. Hence, a small discount rate at the beginning is 'memorized' in the discount factor for all times and, therefore, raises the weight given to the future not only at early times, but also in the long run. However, if the dominance of the discount rate in the strong sustainability scenario keeps up long enough (which is guaranteed by propositions 2 and 3) the discount factors in the strong sustainability scenario will fall below that of

[13]The initial values in the example are $x_1(0) = x_2(0) = 1$.

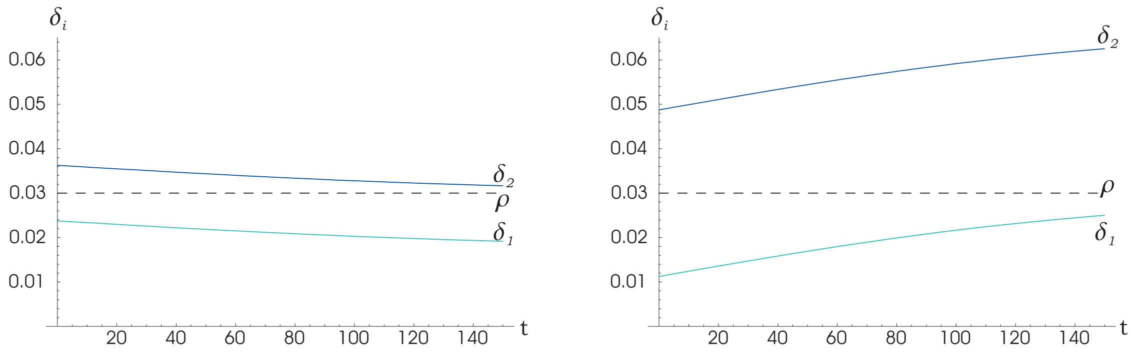


Figure 3.1 : Numerical example for the time development of social discount rates over time in years. The upper line represents the social discount rate for the produced consumption stream, the lower line represents the discount rate for the environmental service stream. The dashed line reflects the pure rate of time preference, corresponding to the common discount rate if perfect substitutability in consumption is assumed. In the left diagram the substitutability parameter is chosen to be $s = .5$, on the right it is $s = -.5$. The other parameters coincide for both scenarios and are $\rho = 3\%$, $\hat{x}_2 - \hat{x}_1 = 2.5\%$ and $a_1 = a_2 = .5$.

the weak sustainability scenario and, thus, give a lower weight to future environmental service streams. In the depicted scenario this would be observed after $t = 195$ years and, sooner or later, it will happen under all parameter specifications as is assured by the following proposition.

Proposition 5: Evaluating the social discount rates for a *given growth scenario* under the assumptions 1 and 2 the following assertion holds.
Assume the existence of $\epsilon > 0$ and $t^* \in [0, \infty)$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all $t \geq t^*$. Then for any $t_0 \in [0, \infty)$ there exists $\bar{t} \in [0, \infty)$ such that $D_i^{\chi^{s<0}}(t, t_0) < D_i^{\chi^{0<s<1}}(t, t_0)$ for all $t > \bar{t}$ and $i \in \{1, 2\}$.

This result clearly counteracts the intuition that environmental goods, which in relative terms become increasingly scarce over time, should be valued higher in the long term, in a setting with strong sustainability preferences and strongly limited substitutability, than in a setting with weak sustainability preference and only moderate limitedness in substitutability.

As discussed in connection with proposition 2 the reason for the time development of the social discount rates is found in the development of the value share. Therefore, the latter should also be the key to understand the supposed puzzle that a strong sustainability decision-maker gives less weight on a long-run environmental service stream

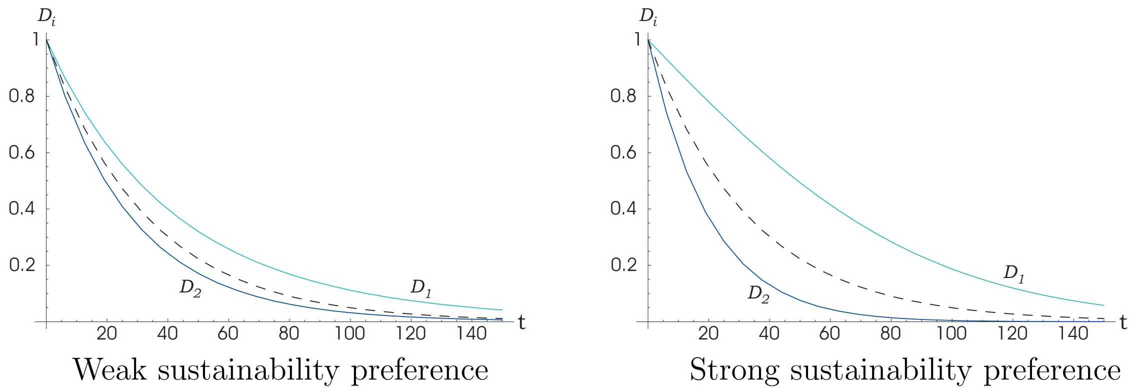
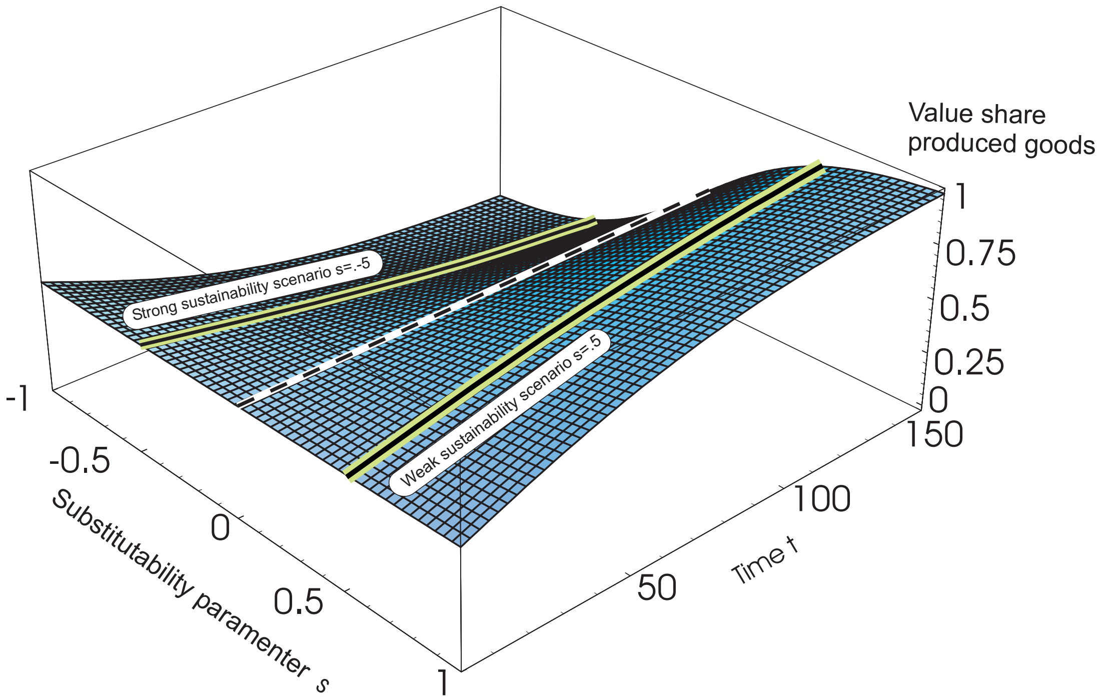


Figure 3.2: Numerical example continued (same specifications as for figure 3.1). Drawn are the social discount factors for the environmental (upper line) and the produced (lower line) good. The dashed line reflects exponential discounting corresponding to the pure rate of time preference.

than does a weak-sustainability decision-maker.[14] The answer to the puzzle turns out to be closely related to an observation by Gerlagh & van der Zwaan (2002). The authors find in a similar stylized growth scenario that under *strong limitedness in substitutability* between the two classes of commodities, the *value share of man-made consumption goes to zero in the long run.* While Gerlagh & van der Zwaan (2002) assume that produced consumption grows over all bounds whereas environmental service streams are bounded, I can show in my setup that this proposition holds true also if consumption in both goods grows without bounds but the produced consumption stream grows at a faster rate (compare proof of proposition 3). The same reasoning holds if both growth rates decline, but the environmental service stream declines at a faster rate. Figure 3.3 depicts how the value share of the produced consumption stream evolves in the scenario underlying figures 3.1 and 3.2. One can clearly observe how the value share of produced consumption grows for a weak sustainability scenario and falls for a strong sustainability scenario. Only for the specification at the border between the two different regions where $s = 0$ the value share stays constant over time (corresponding to the Cobb-Douglas evaluation of proposition 1).

By definition, the value share is a combination of the amount consumed of a consumption stream and its evaluation. In the stylized growth scenario analyzed in this section, the environmental service stream *grows relatively scarce* over time while produced consumption becomes relatively more abundant. At the same time the limited-

[14]This holds as long as the only difference between their preference specifications is the degree of substitutability between the man-made and environmental goods and service flows.



Development of the value share of produced consumption over time

Figure 3.3: Numerical example continued (same specifications as for figure 3.1). Drawn is the value share of the produced consumption stream. The thick lines correspond to the substitutability parameters used for the weak and strong sustainability preference scenario drawn in figures 3.1 and 3.2.

ness in substitutability causes a unit of environmental services to be *increasingly more valuable* than a unit of produced consumption. In the weak sustainability preference scenario (moderate limitedness in substitutability), the (relative) physical scarcity of the environmental service stream dominates, and the value share of the environmental service stream declines, while the value share of the produced consumption stream grows. Limited substitutability and relative scarcity with respect to a consumption stream that increasingly dominates the value share of welfare, causes the discount rate for the environmental service stream to decline.

However, with a strong sustainability preference, corresponding to strong limitedness in substitutability, the increase in unit value dominates the (relative) physical scarcity in determining the value share of the environmental service stream. Therefore not only a unit of environmental service grows more valuable over time, but also the total amount of environmental services consumed grows more valuable than the total amount of produced goods consumed. In consequence, as the value share of the produced consumption stream declines to zero, less and less attention is paid in the social discount rate for the environmental service stream to the limited substitutability with respect to a consumption stream that grows increasingly unimportant for welfare. This effect is best understood when considering the extreme of a strong sustainability preference. For $s \to -\infty$ the evaluation functional converges to $\mathcal{U} = \int_0^\infty \min\{x_1, x_2\} e^{-\rho t}\, dt$. Once the economy is scarcer in environmental service flows than in produced consumption, a decision-maker with these preferences will only pay attention to the environmental service streams. Therefore the time development of his evaluation of an extra unit of environmental services is solely generated by the pure rate of time preference ($\delta_1 = \rho$). *For such a decision maker an increase in relative scarcity of the environmental amenity stream with respect to produced consumption is of no importance* (as soon as the critical level $x_1 = x_2$ has been exceeded). With a growing relative scarcity of the environmental services and a declining value share of the produced services, the other preference specifications in the strong sustainability domain converge towards a similar evaluation. It implies that less attention is paid to the increase in relative scarcity and $\lim_{t\to\infty} \delta_1 = \rho$.

The focus of my analysis has been the time development of the weight given to future consumption streams. Let me point out that social discount rates for the different scenarios at a given point of time reflect well the different concepts of sustainability. At any given point of time, the *difference in evaluation* between an extra unit of *environmental services* and an extra unit of *produced consumption* increases in the *relative scarcity* of the environmental service as well as in the *limitedness in substitutability*. This fact is easily observed by taking the difference between equations (3.6) and (3.3) yielding

$$\delta_2(t) - \delta_1(t) = (1-s)\left(\hat{x}_2(t) - \hat{x}_1(t)\right) . \tag{3.9}$$

This difference in social discount rates in equation (3.9) generates a relative difference in weights given to the consumption streams corresponding to

$$\begin{aligned} \frac{D_1^{\mathbf{X}}(t,t_0)}{D_2^{\mathbf{X}}(t,t_0)} &= \exp\left(-\int_{t_0}^{t} \delta_1(t') - \delta_2(t')\, dt'\right) \\ &= \exp\left(\int_{t_0}^{t} (1-s)\left(\hat{x}_2(t') - \hat{x}_1(t')\right)\, dt'\right) . \end{aligned} \tag{3.10}$$

A stronger notion of sustainability corresponds to a higher limitedness in substitutability $(1-s)$ in the welfare function. As equations (3.9) and (3.10) show, such an increase in $(1-s)$ implies also an increase in the weight given to environmental services as opposed to produced consumption. Moreover, this difference is monotonously growing over time as relative scarcity of the environmental service stream increases. Such a relation reflects well the intuition behind associating the strength of the notion of sustainability with the limitedness of substitutability in the corresponding welfare function.

However, what I have shown in this section, is that a differentiation between a weak and a strong notion of sustainability through an according parametrization of the substitutability between environmental services and produced consumption streams in the welfare function, implies at the same time that a stronger notion of sustainability results in a reduction of the weight given to the future as opposed to the present. Whenever environmental services or both consumption streams are declining over time ($\hat{x}_1 < \min\{\hat{x}_2, 0\}$), such a reduced attention paid to future service and consumption streams seems to oppose the fundamental objective of a sustainable development as expressed in the Brundtland report (compare section 3.1). I want to offer two different perspectives on the derived relationship between the limitedness in substitutability and the accounting weights for the future in the face of the different notions of sustainability.

The first perspective is that, in the analyzed growth scenario, any parameter constellation corresponding to $s < 1$ gives more weight to future environmental services than a welfare function assuming perfect substitutability (which does not pay attention at all to an increase in relative scarcity). Giving a relatively higher weight to the scarcer environmental goods comes at the cost of shifting weight from the future environmental services to the present environmental services.[15] The notions of *strong* versus *weak* sustainability only relate to the substitutability between the different classes of goods and services. When concerned with intertemporal comparisons in a growth scenario as analyzed in this section, applying a weak sustainability preference for project evaluation corresponds to a stronger sustainability demand in the sense that a higher weight is given to long-run future consumption and service streams. The second perspective is that a difference between a weak and a strong notion of sustainability should not only be mapped into a different degree of substitutability between the two classes of goods. It can simultaneously be required that a stronger notion of sustainability should go along with a decrease in intertemporal substitutability. As discussed in connection to equation (2.4) on page 20, a decrease in intertemporal substitutability goes along with an increase in weight given to future consumption streams, whenever growth is declining, and can counteract the effect analyzed above.

[15]Note that this is despite the fact that absolute scarcity and relative scarcity of the environmental service stream increases over time for $\hat{x}_1 < \min\{\hat{x}_2, 0\}$.

Chapter 4

Discounting and Project Evaluation

4.1 Social Discounting in a Cost Benefit Analysis of a Small Project

In this chapter, I elaborate how the social discount rates and factors derived in the previous chapters have to be applied in the evaluation of a small project. This section presents and relates different views on the social discounting and pricing of costs and benefits. In section 4.2, I relate such a social cost benefit analysis to an evaluation in an (imaginary) market system. The project analyzed in this chapter is characterized as a small change $\Delta\mathsf{x}$ of a consumption plan x^0. Exercising the project yields a new consumption and service stream $\mathsf{x}=\mathsf{x}^0 + \Delta\mathsf{x}$ with $x_i = x_i^0(t) + \Delta x_i(t)$, $i \in \{1,2\}$, $t \in [0,T]$. At each point of time $\Delta x_i(t)$ should be small as compared to $x_i(t)$ so that I can expand $U(x_1(t) + \Delta x_1(t), x_2(t) + \Delta x_2(t), t)$ first order in the $\Delta x_i(t)$ (small project assumption). Then, the welfare of the new consumption path can be written as

$$\begin{aligned}
\mathcal{U} &= \int_0^T U(x_1^0(t) + \Delta x_1(t), x_2^0(t) + \Delta x_2(t), t)\, dt \\
&= \int_0^T U(x_1^0(t), x_2^0(t), t) + \frac{\partial U}{\partial x_1}(t)\Delta x_1(t) + \frac{\partial U}{\partial x_2}(t)\Delta x_2(t) + O(\Delta x(t)^2)\, dt \\
&= \mathcal{U}^0 + \int_0^T \frac{\partial U}{\partial x_1}(t)\Delta x_1(t) + \frac{\partial U}{\partial x_2}(t)\Delta x_2(t) + O(\Delta x(t)^2)\, dt\ , \qquad (4.1)
\end{aligned}$$

where the marginal utilities are evaluated along x^0. Equation (4.1) states that neglecting terms of second order in Δx, the project raises welfare, if and only if,

$$\int_0^T \frac{\partial U}{\partial x_1}(t)\Delta x_1(t) + \frac{\partial U}{\partial x_2}(t)\Delta x_2(t)\, dt > 0 \; . \tag{4.2}$$

The integral represents a cost benefit functional in continuous time with valuation derived from the social welfare objective given in equation (2.1). If the path x^0 is optimal, all feasible projects $\Delta\mathsf{x}$ should yield an evaluation smaller or equal to zero. In the following, I derive different perspectives on how to evaluate whether a particular project can increase welfare, and relate them to the results of the preceding chapter. Before doing so, let me assume that at a reference time t_0 there exist prices $p_1(t_0)$ and $p_2(t_0)$ fulfilling $\frac{p_1(t_0)}{p_2(t_0)} = \frac{\frac{\partial U}{\partial x_1}(t_0)}{\frac{\partial U}{\partial x_2}(t_0)}$. In general t_0 will be the present ($t_0 = 0$) and prices $p_1(t_0) = p_1(0)$ and $p_2(t_0) = p_2(0)$ are either market prices, if present markets exist, or, more likely for the environmental service streams, they are prices derived from direct and indirect methods of evaluation, like for example contingent valuation or hedonic price studies (see e.g. Hanley, Shogren & White 1997, 383 et sqq., or Mäler & Vincent 2005). In equation (4.2) time specific marginal utilities are used to evaluate the changes in x_1 and x_2 at every point of time. By relating the marginal utilities in equation (4.2) for different points of time with the help of equation (2.2), I arrive at the perspective of social discounting as it was adopted in the previous chapters. Together with the above relation for prices at t_0 I obtain the form

$$\int_0^T D_1^{\mathsf{x}^0}(t,t_0)p_1(t_0)\Delta x_1(t) + D_2^{\mathsf{x}^0}(t,t_0)p_2(t_0)\Delta x_2(t)\, dt > 0 \; . \tag{4.3}$$

Equation (4.3) takes the prices at t_0, usually the present, to determine the relative value of x_1 and x_2 at t_0 and propagates both prices over time by means of the marginal utility propagators $D_1^{\mathsf{x}}(t,t_0)$ and $D_2^{\mathsf{x}}(t,t_0)$ respectively. The prices $D_i^{\mathsf{x}}(t,t_0)\, p_i(t_0)$ could be referred to as social accounting prices.[1] Another interpretation is to take the factors $D_i^{\mathsf{x}}(t,t_0)$ as *good–specific social discount factors*. This view corresponds to the analysis of chapters 2 and 3. Applying equation (4.3) to the growth scenario in section 3.3 with a weak sustainability preference would imply a marked up and falling discount rate for the produced consumption stream x_2 and marked down and falling discount rate for the environmental service stream. It is important to be aware that *either* one can argue that prices of the environmental service stream rise due to its increasing relative scarcity, *or*

[1] Note that these prices $D_i^{\mathsf{x}}(t,t_0)\, p_i(t_0)$, in general, do not coincide with the capital measured market prices that will be studied in section 4.2.

one can apply the good-specific discount rates discussed earlier. Doing both at the same time yields a wrong evaluation.

An interesting special case is the evaluation of a project that affects only consumption of the environmental service streams at different points of time ($\Delta \mathsf{x}_2 = 0$). Then (4.3) is equivalent to

$$\int_0^T D_1^{\mathsf{x}^0}(t,t_0)\Delta x_1(t)\,dt > 0\,.$$

An important consequence of the discussion in chapter 3.3 is the following. Considering a *partial model of the environmental sector*, optimal discounting can be *hyperbolic and time consistent* with a marked down discount rate. Moreover, for the evaluation of such a project, the *relative* weight given to environmental services as opposed to produced consumption is of no importance. Therefore it appears particularly catchy that an evaluation based on a strong sustainability preference - in the sense and scenario of the last section - gives less weight to long-term environmental service flows than an evaluation based on a weak sustainability preference.

By factoring out $D_1^{\mathsf{x}^0}(t,t_0)$ or $D_2^{\mathsf{x}^0}(t,t_0)$ in equation (4.3) the evaluation functional becomes

$$\int_0^T \left[p_1(t_0)\Delta x_1(t) + \frac{D_2^{\mathsf{x}^0}(t,t_0)}{D_1^{\mathsf{x}^0}(t,t_0)} p_2(t_0)\Delta x_2(t) \right] D_1^{\mathsf{x}^0}(t,t_0)\,dt > 0 \quad \text{or} \tag{4.4}$$

$$\int_0^T \left[\frac{D_1^{\mathsf{x}^0}(t,t_0)}{D_2^{\mathsf{x}^0}(t,t_0)} p_1(t_0)\Delta x_1(t) + p_2(t_0)\Delta x_2(t) \right] D_2^{\mathsf{x}^0}(t,t_0)\,dt > 0\,. \tag{4.5}$$

Equations (4.4) and (4.5) take the more usual view, that there is one common discount rate applicable to all goods (*the* discount rate). Equation (4.4) can be interpreted the following way. Let the evaluation start out with the prices in $t_0 = 0$. Then the first good is taken to be the *numeraire* (in the sense of keeping its price constant). Hence, the change of marginal utility of the first good expressed by $D_1^{\mathsf{x}^0}(t,t_0)$ becomes *the* discount factor and the (contemporaneous) value of the second good must be propagated by the relative change of marginal utility of good two relative to good one, i.e. by $\frac{D_2^{\mathsf{x}^0}(t,t_0)}{D_1^{\mathsf{x}^0}(t,t_0)}$. Applied again to the setup of chapter 3.2 with the example of moderately limited substitutability between environmental and produced consumption streams, the social discount rate for the environmental service stream would be *the* discount rate and discounting would take place with the lower hyperbolic discount rate δ_1.

A more common perspective on cost benefit analysis corresponds to equation (4.5) which is the analogue taking x_2 to be the numeraire. Such a cost benefit evaluation takes the social discount rate of produced consumption as *the* discount rate. Time development of the accounting price for the environmental service stream is characterized by the expression $\frac{D_1^{\mathbf{X}^0}(t,t_0)}{D_2^{\mathbf{X}^0}(t,t_0)}$. Let me normalize $p_2(0)$ to unity and define $p_1^*(0) = \frac{\frac{\partial U}{\partial x_1}(0)}{\frac{\partial U}{\partial x_2}(0)}$ as the value of a unit of environmental services in units of produced goods in the present. Then, choosing $t_0 = 0$, equation (4.5) together with equation (3.10) imply that for the scenario analyzed in chapter 3.3 the (relative) pricing of the environmental service stream in units of produced consumption develops as:

$$p_1^*(t) = p_1^*(0) \exp\left(\int_0^t (1-s)\left(\hat{x}_2(t') - \hat{x}_1(t')\right) dt'\right) . \tag{4.6}$$

This way of setting up a cost benefit evaluation is also promoted by Arrow et al. (1995). From this perspective the authors criticize Weitzman's (1994) derivation of a marked down hyperbolic 'environmental discount rate' as not properly converting environmental benefits into a flow of (produced[2]) consumption equivalents. Arrow et al. (1995, 139) state that the "essence of social discounting is to convert all effects into their consumption equivalents and then to discount the resulting stream of consumption equivalents at the social rate of time preference. Incorporating environmental effects does not change the discount rate itself but does require special attention to the proper relative pricing of environmental goods over time". Note that Arrow et al. (1995) use the term 'social rate of time preference' for the social discount rate δ in the sense of the one commodity equation (2.4).

In fact Weitzman (1994) neither models the environmental good explicitly, nor does he state a functional form for the preferences. Also does he not account separately for environmental changes and growth of produced consumption. Instead, Weitzman derives an overall discount rate under the assumption that produced consumption is growing at the cost of degrading the environment.[3] The assumed functional form for this relationship renders an overall discount rate that is smaller than in the absence of environmental externalities and falling over time. As this so called 'environmental discount rate' does not explicitly distinguish between the value development of environ-

[2]Arrow et al. (1995) use the one commodity equation (2.4) as point of departure for their discussion on discounting. In the section on 'the environmental and discounting' the authors make the point that environmental benefits have to be converted into consumption equivalents. In the view of my explicit two commodity setup, I identify environmental benefits as environmental service and consumption streams x_1 and their (obviously non-environmental) consumption equivalents as corresponding to my produced consumption stream x_2.

[3]An alternative interpretation offered by Weitzman is that the environment is a luxury good whose demand grows over time.

mental service streams and that of produced consumption it could at most be applied to a project where the assumed fixed relationship between production growth and environmental decline holds. More generally Groom et al. (2005, 458) criticize that “in many ways Weitzman’s environmental discount rate is difficult to interpret in light of the reduced form set up and, in particular, the absence of an explicit modeling of preferences, environmental goods and externalities.”

Arrow et al. (1995, 140) continue that when properly converting into consumption equivalents, the environmental considerations do “not change the discount rate to apply to the consumption stream”. Now equation (4.5) gives a cost benefit analysis in the perspective of Arrow et al. (1995) and equation (4.6) works out how for the scenario in chapter 3.3 a ‘proper relative pricing of environmental goods over time’ converts the environmental service and amenity stream into (produced) consumption equivalents. However, in the same scenario it was also observed that under limited substitutability in consumption *the increasing relative scarcity of the environmental service stream also changes the discount rate* δ_2 *that has to be applied to the (produced) consumption stream* x_2. In particular under the assumption of a moderate limitedness in substitutability such an environmental consideration still can give rise to hyperbolic discount rates for the produced consumption stream. However, the produced consumption stream gets a mark up and not a mark down. Moreover, under the assumption of strong limitedness in substitutability, a growing relative scarcity in the environmental service stream can result in a growing value share of the environmental goods and go along with an increasing discount rate.

4.2 Relation to a Complete Market Evaluation

After having worked out the evaluative structure for the setting of incomplete future markets, I want to point out how it relates to a scenario where markets are complete and evaluation is reflected in the corresponding market prices. The prices are derived by setting up the budget constraint of a representative consumer. Welfare is again assumed to be of the general form of equation (2.1) though restricted by the assumptions that follow below. For this section I will assume that the social optimum can be decentralized by an appropriate price system. Prices are measured in units of capital which can be regarded either as money or as real capital. These current value prices are denoted by $p_1(t)$ and $p_2(t)$. The interest rate on capital is $r(t)$. Remuneration for a fixed offer of one unit of labor $w(t)$ is only introduced for ‘completeness’ of the budget constraint. All these variables are exogenous to the representative consumer. His choice is between saving $\dot{k}(t)$ units of the capital good k and consuming the amounts $x_1(t)$ and $x_2(t)$. For x_1 being essential life support services provided by the environment, such an immediate choice of a representative agent is of course fictitious. For environmental goods like a hiking trip or a scenic view one might get closer to existing future markets. However, the setting is only meant to relate discounting in markets, or with respect to market based prices, to a social cost benefit setup. With the above assumptions the budget constraint of the representative agent is given by the equation

$$\dot{k}(t) = r(t)k(t) + w(t) - p_1(t)x_1(t) - p_2(t)x_2(t) \ .$$

Together with equation (2.1) it follows that the Hamiltonian describing the optimization problem of the representative agent is

$$H = U\left(x_1(t), x_2(t), t\right) + \lambda(t)\ \left[r(t)k(t) + w(t) - p_1(t)x_1(t) - p_2(t)x_2(t)\right] \ .$$

In the following, I assume that a sufficiency condition for the optimization problem is met.[4] Moreover, I assume a continuous control (consumption) path and an interior solution. The solution for the consumption path is denoted by $\mathfrak{x}$. Along this path the following necessary conditions for an optimum must be satisfied:

$$\frac{\partial H}{\partial x_1} = \frac{\partial U}{\partial x_1} - \lambda(t)\, p_1(t) \stackrel{!}{=} 0 \,, \tag{4.7}$$

$$\frac{\partial H}{\partial x_2} = \frac{\partial U}{\partial x_2} - \lambda(t)\, p_2(t) \stackrel{!}{=} 0 \,, \tag{4.8}$$

$$\frac{\partial H}{\partial k} = \lambda(t)\, r(t) \stackrel{!}{=} -\dot{\lambda}(t) \ . \tag{4.9}$$

From equations (4.7) and (4.8) I obtain the relations:

$$\frac{\frac{\partial U}{\partial x_1}(t)}{\frac{\partial U}{\partial x_2}(t)} = \frac{p_1(t)}{p_2(t)} \qquad \text{and}$$

$$\frac{\frac{\partial U}{\partial x_i}(t)}{\frac{\partial U}{\partial x_i}(t_0)} = \frac{\lambda(t)}{\lambda(t_0)} \frac{p_i(t)}{p_i(t_0)} \quad i \in \{1, 2\} \ . \tag{4.10}$$

Integration of equation (4.9) yields the shadow price of capital

$$\lambda(t) = c e^{-\int_0^t r(t')dt'} \quad \text{with the integration constant } \lambda(0) = c \in \mathbb{R}_+ \ . \tag{4.11}$$

Analogous to the social discount factors describing marginal utility propagation on the preference side, let me define the *time propagator of capital* as

$$R(t_0, t) = e^{\int_{t_0}^t r(t')dt'} \ .$$

[4]See Takayama (1994, 660 sqq.), Chiang (1992, 214 et sqq.) and Seierstad & Sydsaeter (1977) for different sufficiency conditions.

It describes how much capital in t can be derived from an extra unit of capital in t_0. Note that analogously to the reasoning in chapter 2.2 on the relation between δ_i and D_i^{X}, the productivity of capital $r(t)$ can be interpreted as the *generator of capital propagation*. I have defined $R(t_0, t)$ in a way that again $R(t, t_0) = \frac{1}{R(t_0,t)} = e^{-\int_{t_0}^{t} r(t')dt'}$ is the factor which is *dis*counting with capital productivity. I refer to $R(t, t_0) = \frac{1}{R(t_0,t)}$ as the inverse capital propagator. Equation (4.11) shows that the shadow value of capital at time t is inversely proportional to the productivity of capital between the present and time t, i.e. $\lambda(t) \propto R(t, t_0)$.[5] This relation is straight forward as a unit of capital today can be turned into $R(t_0, t)$ units of capital in period t. Therefore a unit of capital in time t is worth $\frac{1}{R(t_0,t)} = R(t, t_0)$ units of capital today.

Inserting $R(t_0, t)$ into equation (4.10) the following relation between the time propagator of marginal utility $D_i^{\mathsf{X}}(t, t_0)$ of good i, the capital propagator and the price of good i is obtained:

$$p_i(t) = D_i^{\mathsf{X}}(t, t_0)\, p_i(t_0)\, R(t_0, t)\,. \tag{4.12}$$

Equation (4.12) shows that time development of (capital measured) prices depends on two influencing factors. One is the effect discussed in the previous chapters depending on the change of marginal utility expressed by $D_i^{\mathsf{X}}(t, t_0)$. In addition, the *current value prices* $p_i(t_0)$ and $p_i(t)$ corresponding to different periods have to be related. As prices of the goods are measured in units of the capital good, this is achieved by the capital propagator $R(t_0, t)$.

For the one commodity setting it is often assumed that capital is measured in units of consumption (e.g. Barro & Sala-i-Martin 1995, 62). Similarly, I could assume in the two commodity setting that capital is measured in units of produced consumption. This assumption makes the current value price of produced consumption constant over time.[6] Therefore equation (4.12) implies for $i = 2$ that the inverse capital propagator $R(t, t_0)$ and the propagator of marginal utility $D_2^{\mathsf{X}}(t, t_0)$ coincide. This is because capital measured in units of produced consumption now reflects the value development of produced consumption over time. With regard to the non-constancy of the social discount rates in the scenario of chapter 3.3, note that such a measurement of capital implies that $r(t)$ exhibits the same non-constant form as derived for δ_2.

With this background, let me finally analyze how the evaluation of the small project

[5] The shadow value reflects the value of an extra unit of capital in units of welfare along the optimal path. For a closer discussion and the derivation of this interpretation of a shadow price (costate variable) compare e.g. Kamien & Schwartz (2000, 136 et sqq.). Note that λ is the present value shadow price.

[6] Then the current value price of produced consumption is measured in units proportional to itself.

from the preceding section would be evaluated if complete markets existed for all times.[7] Applying equation (4.12) to equation (4.3), the following evaluation functional for the project is obtained:

$$\int_0^T [p_1(t)\Delta x_1(t) + p_2(t)\Delta x_2(t)]\, R(t,t_0)\, dt > 0 \ . \tag{4.13}$$

This time the social discount factors $D_i^{\chi}(t,t_0)$ are not needed for evaluation. The price development accounts already for the change in welfare. But as prices are measured in capital, and, if capital is productive, the present value of a unit of capital in the future is less than the value of a unit of capital in the present, the future prices have to be *dis*counted with capital productivity. Therefore capital productivity can be regarded as *the* common discount rate for both goods. If capital is measured in terms of produced consumption, and thus $R(t,t_0) = D_2^{\chi}(t,t_0)$, equation (4.13) coincides with equation (4.5).

4.3 Summary

To evaluate long–term projects, an expression for the development of valuation over time is needed. Social discount rates represent such an expression. They allow the economist to think in rates and elasticities, and lay out different contributions in a nice additive form. The study in part I of this dissertation has worked out one such contribution to value development over time, which emerges in a multi-commodity world with limited substitutability between different forms of consumption. I have analyzed a scenario, in which produced consumption is assumed to grow at a faster rate than environmental services. In this setting, I identified moderate and strong limitedness in substitutability in the welfare function with the notions of weak and strong sustainability on sides of the decision maker. I have derived that, under the assumption of moderate limitedness in substitutability, the social discount rates for (both) future consumption streams fall over time. On the other hand, the assumption of strong limitedness in substitutability goes along with rising social discount rates. This result was related to Gerlagh & van der Zwaan's (2002) finding that under strong limitedness in substitutability the value share of produced consumption falls to zero, even when its physical share grows to one. Such

[7] Having assumed that the social optimum can be decentralized in a complete market system, such an evaluation is only of theoretical interest to compare the resulting cost benefit functional to that of section 4.1.

a development goes along with a reduced attention paid to the increase in relative scarcity of the environmental service flow over time. In consequence, less weight is given to environmental services in the long run than under the assumption of only moderate limitedness in substitutability.

In such a scenario, the identification of the strength of the notion of sustainability, with the substitutability between the two classes of service and consumption streams, seems puzzling when environmental services also decline in absolute terms. Then, a notion of stronger sustainability delivers a weaker commitment to a sustainable development in the sense of valuing future resources. If such a relationship is unwanted, it has been suggested that a stronger notion of sustainability can be translated simultaneously into a higher limitedness of substitutability between environmental service streams and produced consumption, and into a reduction of intertemporal substitutability. I have elaborated how the derived discount rates and factors have to be applied in project evaluation. Either, they can be used to directly propagate individual prices over time, or a numeraire has to be chosen. In the latter case, the discount rate of the numeraire becomes *the* discount rate, and other consumption streams have to be converted into contemporaneous equivalents of the numeraire. However, not only magnitude, but also the form of discounting depend on the choice of the numeraire and its substitutability to other goods.

Part II

Intertemporal Risk Aversion and the Precautionary Principle

Chapter 5

Preliminaries

5.1 Introduction

Recently Hahn & Sunstein (2005, 1) predicted in the Economists' Voice that "Over the coming decades, the increasingly popular 'precautionary principle' is likely to have a significant impact on policies all over the world." However, there is an ongoing debate between and among economists, environmental scientists and policy makers about the merit and meaning of the precautionary principle. The usual formulations of the principle are generally vague, as discussed for example in Turner & Hartzell (2004) and Sandin (2004). In part II of my dissertation I suggest an axiomatic formalization of decision-making under uncertainty that takes up an important concern of the precautionary principle, related to the willingness to undergo preventive action in order to avoid a threat of harm. For doing so, I introduce a new notion of risk aversion in the multi-commodity case and connect it to the idea of precaution.

A second point of view, motivating the line of thought of this study, is the following. I derive the general time consistent model, satisfying the von Neumann & Morgenstern (1944) axioms for an individual period, and additive separability over time when restricted to certain outcomes. I consider this question particularly interesting as these are the two predominantly used specifications in the respective framing scenarios of atemporal choice under uncertainty and intertemporal choice under certainty. Merging the assumptions underlying the respective representations into a common, time consistent framework, does not result in intertemporally additive expected utility, but a more general class of representations that also accommodate precautionary decision rules. In this framework, I analyze how Epstein & Zin's (1989) disentanglement between risk aversion and intertemporal substitutability extends to a multi-commodity setting. I show that in a world with many commodities absolute values of risk aversion and intertemporal substitutability are good-dependent, while a particular relation between the two is invariant. This invariant gives rise to the concept of intertemporal risk aversion. In opposite to the extension of standard risk aversion to a multi-commodity setting as developed by Kihlstrom & Mirman (1974), the concept of intertemporal risk aversion is not confined to comparisons of ordinally equivalent preferences.[1]

My setting closely relates to the seminal work of Kreps & Porteus (1978), who extend the atemporal von Neumann-Morgenstern setting for choice under uncertainty to a temporal structure. Under the assumption of intertemporal consistency, the authors obtain a recursive representation that uses expected utility evaluation within each period, and a generally nonlinear time aggregation from one period to the next. Kreps and Porteus show that an agent behaving in accordance with their axioms generally exhibits a preference for the timing of risk resolution. The representation brought forward by the authors can be understood as an extension of Koopmans's (1960) recursive utility model under certainty to a recursive model for risky settings. My study shows that even when starting from a time-additive model for certain outcomes, the general time consistent model for the evaluation of risky outcomes will exhibit recursivity and preference for the timing of risk resolution. While Kreps & Porteus' (1978) representation is more general, my model gives rise to an attractive structure that enhances the economic interpretation. This is achieved not only by introducing and relating measures of atemporal risk aversion, intertemporal substitutability and intertemporal risk aversion, but also by the following reasoning. Contributing to the intricacy of interpreting Kreps & Porteus' (1978) representation is the fact that it crucially depends on a nonlinear aggregation of utility over time. In my view, working with a utility (or welfare) function that is additive over time on certain outcomes greatly simplifies the interpretation and, thus, the move from mathematical representation to economic intuition.[2]

[1] In this respect it stands closer to the application of the theory of risk aversion to the indirect utility function as brought forward by Stiglitz (1969). For fixed prices, Stiglitz applies the standard Arrow-Pratt approach of risk aversion for one commodity to income. However this approach is constraint to compare lotteries along an individual's Engel curves in a market environment.

[2] In such an intertemporally additive representation, a welfare gain of one unit today and a welfare gain of another unit in the next period is just as good as a welfare gain of two units in a third period. Such a reasoning generally is wrong in the representation of Kreps & Porteus (1978). Note that in a setting with stationary preference and a positive rate of time preference, the latter of course has to be integrated into the interpretation. This fact, however, is not related to the nonlinearity of utility aggregation over time as found in the representation of Kreps & Porteus (1978).

Epstein & Zin (1989) analyze Kreps & Porteus' (1978) representation[3] in a one commodity setting in order to disentangle information about the attitude towards risk and towards intertemporal substitutability. Such a distinction between risk aversion and intertemporal substitutability is not possible within a standard intertemporally additive expected utility model.[4] Using a representation that allows for different choices of Bernoulli utility[5], I analyze to what extend such a specification of risk aversion and intertemporal substitutability proves useful when studying a multi-commodity world. I work out that there is no longer a canonical measure of risk aversion or intertemporal substitutability, as both of these quantities generally vary between different goods. This variation between different goods can be expressed as a dependence on the choice of the particular representing Bernoulli utility function (for given preferences). I identify a relation between the characterizations of risk aversion and intertemporal substitutability that is invariant under different choices of Bernoulli utility. It is this invariant quantity that gives rise to the notion of intertemporal risk aversion. An axiomatic formalization of the latter concept is developed and quantitative measures are worked out.

Later, in part III of the dissertation I establish a general relation between the measures of risk aversion and intertemporal substitutability and Kreps & Porteus' (1978) preference for the timing of uncertainty resolution. This relation answers a question raised by Epstein & Zin (1989, 952 et seq.) on the interlacement of (standard) risk aversion, intertemporal substitutability and the preference for the timing of uncertainty resolution. My representation suggests that the concept of intertemporal risk aversion is also at the basis of a preference for the timing of uncertainty resolution. Seeking for reasonable simplifications of the model structure, I analyze the consequences of indifference to the timing of uncertainty resolution and of stationarity with respect to risk evaluation. Indifference to the timing of uncertainty resolution yields a representation that captures intertemporal risk aversion in a single parameter. Moreover, it simplifies the model structure by allowing for a non-recursive description of lotteries. In particular, such a representation allows to disentangle risk aversion from intertemporal substitutability

[3]The precise difference of Epstein & Zin's (1989) setting compared to Kreps & Porteus (1978) is that the first only analyze a one commodity setting with no history dependence and a time aggregation that exhibits constant elasticity of substitution. However, they allow for a more general evaluation of uncertainty than that implied by the von Neumann-Morgenstern axioms and, moreover, extend the framework to allow for an infinite time horizon.

[4]In the intertemporally additive expected utility model, the elasticity of intertemporal substitution is confined to the inverse of the Arrow-Pratt measure of relative risk aversion (Weil 1990).

[5]Bernoulli utility describes a cardinal function, that, by itself represents choice of certain one period outcomes. In combination with uncertainty evaluation functionals and time aggregation rules it serves as the basis for more general evaluation.

when evaluating lotteries over consumption paths non-recursively. Based on a study by Chew & Epstein (1990) for homothetic preferences, the latter combination of attributes has been believed to be unfeasible. Moreover, I work out two different axioms of risk stationarity. One yields stationarity of the functionals evaluating uncertainty in every period. The other is a more natural extension of certainty stationarity in a finite time framework. In combination with indifference to the timing of uncertainty resolution, the latter will have strong implications for the choice of the rate of pure time preference.

Part II of my dissertation is structured as follows. In the upcoming section 5.2, I introduce the precautionary principle and motivate my approach to a formalization. Related literature on modeling the concept of precaution and choice under uncertainty is briefly summarized. Section 5.3 formally introduces the concept of general and precautionary uncertainty aggregation rules, which are closely related to generalized means. Chapter 6 develops an axiomatic representation of preferences, based on these uncertainty aggregation rules and a closely related intertemporal aggregation rule. Chapter 6.1 revisits the atemporal von Neumann-Morgenstern setting. Special attention is payed to different possibilities of fixing (gauging) the Bernoulli utility function over the certain outcomes and its consequence for the applicable uncertainty aggregation rule. Chapter 6.2 takes a brief look at the other framing scenario, i.e. additively separable preferences over certain consumption paths, and introduces the concept of an intertemporal aggregation rule. The two framing scenarios are united in chapter 6.3 by a representation for the simplest setting that is sufficiently rich to discuss most of the topics pointed out earlier in this introduction. It consists of a two period framework, where the first period outcomes are certain and the second period outcomes are uncertain. The crucial point of this representation is that it leaves some freedom for the choice of the representing Bernoulli utility function. Chapter 6.4 points out how fixing (gauging) Bernoulli utility in different ways leads to different representations found in the literature.

Chapter 7 discusses the economic content of the representation. In chapter 7.1, I analyze how Epstein & Zin's (1989) distinction between intertemporal substitutability and risk aversion in the one-commodity case carries over to a multi-commodity setting. In particular, I show that in the multi-commodity setting only a particular relation between the two is invariant over different commodities. Chapter 7.2 axiomatically identifies this invariant quantity as a notion of risk aversion itself. Due to its crucial dependence on the intertemporal structure of preference, I call this notion of uncertainty attitude 'intertemporal risk aversion'. Chapter 7.3 shows, how the concept of precaution as it is motivated in the next two sections, coincides with the concept of strict intertemporal risk aversion. In this connection welfare is interpreted as a Bernoulli utility function that exhibits additive value aggregation over time. With such a notion of welfare, intertemporal risk aversion and precaution are reconsidered as risk aversion on welfare. Finally, chapter 7.4 works out quantitative measures of intertemporal risk aversion. Part III of this dissertation is an extension of the analysis carried out in part II. Chapter 8 extends the preference representation and the notion of intertemporal risk aversion to a general non-stationary multiperiod setting. In chapter 9, I introduce two different stationarity conditions for preferences over uncertain outcomes (in the setting with a finite time horizon). Finally, chapter 10 is dedicated to different aspects of preferences for the timing of uncertainty resolution in the sense of Kreps & Porteus (1978), including its relation to standard risk aversion, intertemporal substitutability, intertemporal risk aversion and the pure rate of time preference.

5.2 The Precautionary Principle

The most frequently cited definition of the precautionary principle was agreed upon at the Wingspread Conference in 1998 by 32 participants with different academic and professional backgrounds. The latter state that "it is necessary to implement the Precautionary Principle: *Where an activity raises threats of harm to the environment or human health, precautionary measures should be taken even if some cause and effect relationships are not fully established scientifically*" (Raffensperger & Tickner 1999, 8)[6].

There is some consensus that the precautionary principle emerged as an explicit and (somewhat) coherent principle in the field of environmental policy in the seventies in relation with the German 'waldsterben' (forest dieback) and its possible causes (see Harremoes et al. 2001, 13).[7] Its first major international advocacy came with the adoption of the World Charter for Nature by the General Assembly of the United Nations in 1982.[8] With the decision on phasing out the production of a number of substances *believed* to be responsible for the depletion of the stratospheric ozone layer, the precautionary principle first entered into an international treaty in the Montreal Ozone Layer Protocol in

[6]Emphasis added, the Wingspread declaration is also found online at http://www.gdrc.org/u-gov/precaution-3.html.

[7]However, some authors date the precautionary priciple back to earlier times, cf. Sandin (2004, 462). For example, Martin (1997, 276) writes that "Unambiguous reference to precaution as a management guideline is found in the millennial old tradition of Indigenous People of Eurasia, Africa, the Americas, Oceania, and Australia".

[8]Section (11b) of the World Charter of Nature states that "where potential adverse effects are not fully understood, the activities should not proceed". The Charter is available online at http://www.un.org/documents/ga/res/37/a37r007.htm .

1987. Since then, it has been taken up in a series of international treaties and declarations. Most importantly these include the Third North Sea Conference (1990) with a particularly strong version,[9] the 1992 Rio Declaration on Environment and Development & the Framework Convention on Climate Change,[10] and the Cartagena Protocol on Biosafety (2000)[11]. In the contracts of the European Union, the Precautionary Principle entered into the Maastricht Treaty in 1994, was worked out in a Communication on the Precautionary Principle in 2000 and made it into the European Constitution, which was signed in 2004, but not ratified yet.[12]

The different formulations of the precautionary principle account for a wide range of specifications, reaching from comparatively moderate formulations as in the Rio Declaration on Environment and Development 1992, Principle 15: "Where there are threats of serious or irreversible damage, lack of scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation"[13] to much stronger statements as in the Third North Sea Conference (1990): "apply the precautionary principle, that is to take action to avoid potentially damaging impacts of substances that are persistent, toxic, and liable to bioaccumulate even where there is *no* scientific evidence to prove a causal link between emissions and effects" (Harremoes et al. 2001, 14, emphasis added). Most importantly, however, the formulations are generally vague. A detailed analysis of this problem can be found in Sandin (2004) and Turner & Hartzell (2004). This vagueness is a major source of criticism with respect to the precautionary principle, as prominently expressed by Hahn & Sunstein (2005). The authors argue that "the precautionary principle does not help individuals or nations make difficult choices in a non-arbitrary way. Taken seriously, it can be paralyzing, pro-

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[9]See next paragraph for the formulation.

[10]The wording of the United Nations Framework Convention on Climate Change Article 3.3 is similar to that of the 1992 Rio Declaration on Environment and Development cited in the next paragraph. The Convention is available online at http://unfccc.int/resource/docs/convkp/conveng.pdf .

[11]The preamble directly refers to the precautionary principle as formulated in the 1992 Rio Declaration on Environment and Development, and articles 10 and 11 further elaborate the principle. The Cartagena Protocal on Biosafety is available online at http://www.biodiv.org/biosafety/protocol.asp .

[12]Article III-233 of the draft Treaty establishing a constitution for Europe stipulates: "Union policy on the environment shall aim at a high level of protection taking into account the diversity of situations in the various regions of the Union. It shall be based on the precautionary principle and on the principles that preventive action should be taken, that environmental damage should as a priority be rectified at source and that the polluter should pay."

[13]Compare Rao (2000, 11 et seq.) or find the complete declaration online at http://www.unep.org/Documents.multilingual/ Default.asp?DocumentID=78&ArticleID=1163.

viding no direction at all" (Hahn & Sunstein 2005, 1). They continue that "In contrast, balancing costs against benefits can offer the foundation of a principled approach for making difficult decisions" (Hahn & Sunstein 2005, 1). My goal is to work out such a principled approach for balancing costs and benefits in a non-arbitrary way, which, at the same time, takes up some of the concern of the precautionary principle.

To *motivate my approach*, let me come back to the above mentioned Wingspread definition of the precautionary principle. It requires that "Where an activity raises threats of harm to the environment or human health, precautionary measures should be taken even if some cause and effect relationships are not fully established scientifically" (Raffensperger & Tickner 1999, 8). Yet any reasonable economic model depicting uncertainty will take into consideration a *threat of harm* to human welfare. In a standard model such a threat would be represented by a positive probability of yielding low welfare. Such a probability does not have to be objective[14] and, thus, does not have to be based on a complete scientific understanding. A threat of harm in this sense obviously reduces the expected welfare. A *precautionary measure*, in the sense of the Wingspread definition, is an action taken to avoid such a threat of harm. Hence, it has to take place before the observed impact on welfare. Therefore a formal model, depicting the decision problem at hand, has to consider at least two periods. In addition, the second period has to account for uncertainty. Let me lay out my formal intuition of precaution in such a *simple model*.

Let me denote outcomes in the first and the second period by x_1 and x_2 respectively. For the purpose of this introduction, think of outcomes as a description of consumption, effort and harm for a particular state of the world that a decision maker envisions within a period. Effort accounts for the various endeavors which are undergone in the first period, in order to avoid a threat of harm in the second period. In the following x_1 is assumed to vary in the amount of effort undergone. Of course such an effort can go along with a reduction in consumption. Increasing effort is assumed to reduce welfare within the first period. For the second period uncertainty prevails. Assume that only two outcomes are perceived possible. One is a standard or 'unharmed' outcome, denoted by $\bar{x}_2$, and the other is an outcome, where society suffers serious harm and is denoted by $\underline{x}_2$. Furthermore, let each of the two possible second period outcomes be associated with probabilities $p(\bar{x}_2)$ and $p(\underline{x}_2) = 1 - p(\bar{x}_2)$. I characterize society's welfare (or an individual's utility) by a welfare function u. Obviously welfare is assumed to be

[14]The empirical definitions of probability by frequency or symmetry are usually referred to as objective probabilities. In the situation described by the Wingspread declaration an epistemic approach to probabilities better fits the situation. Here, probabilities are seen as elements of a (non-binary) logic or as beliefs. See the discussion on page 63.

higher in the unharmed state than when society is suffering from serious harm, implying $u(\bar{x}_2) > u(\underline{x}_2)$. The standard evaluation of such a scenario would be depicted by the following equation:

$$u(x_1) + p(\bar{x}_2)u(\bar{x}_2) + p(\underline{x}_2)u(\underline{x}_2) = u(x_1) + \mathrm{E}_p\, u(x_2)\,. \tag{5.1}$$

For simplicity, I assume a stationary welfare function and set the rate of pure time preference to zero.[15] Equation (5.1) is the evaluation rule corresponding to a maximizer of intertemporally additive expected utility (or welfare). Note that I identify evaluation rules with corresponding decision rules by the assumption of welfare (or utility) maximization on some set of feasible outcomes. Moreover, for the remainder of this chapter, the terms welfare and utility will be used synonymously.[16] The evaluation in equation (5.1) translated into real terms is also the standard cost benefit analysis answer for a two period setting with uncertainty, see e.g Brent (1996, 167 et seq.) and Johansson (1993, 142 et seq.). In equation (5.1) the threat of harm $p(\underline{x}_2)u(\underline{x}_2)$ diminishes overall welfare. In consequence, there will always be some willingness on behalf of the decision maker to undergo efforts that decrease or prevent the threat of harm $p(\underline{x}_2)u(\underline{x}_2)$. In accordance with the Wingspread definition, precautionary measures have to be taken in the first period, in order to reduce or eliminate the threat of harm in the second period. If these measures would come at no cost, they would obviously be carried out. The interesting scenario is when such precautionary measures lower the welfare in the first period. A decision maker using equation (5.1) for his evaluation is willing to accept a *reduction of first period welfare* of up to $u(\bar{x}_2) - \mathrm{E}\, u(x_2)$, in order to eliminate the threat of harm. In words, the maximum of first period welfare reduction accepted corresponds to the difference between the welfare derived from the unharmed outcome and the expected welfare when facing the threat of harm.

Yet, this effort *does not seem to suffice* the advocates of the precautionary principle. The authors of the Wingspread declaration state explicitly that "We believe existing environmental regulations and other decisions, particularly those based on risk assessment, have failed to adequately protect human health and the environment, as well

[15] As will be discussed later on, neither a positive rate of time preference nor a non-stationary welfare function change the general insight. Obviously, the cost-benefit approaches mentioned below do apply a positive rate of discount striving for numerical results.

[16] While utility rather alludes to an agent making private decisions, the word welfare suits better to a decision maker in public policy. However, the reasoning carried out in my study is suitable for both scenarios. Moreover, some of the concepts from decision theory discussed in this and the next sections are conventionally labeled in terms of utility rather than welfare, like, for example, expected versus non-expected utility theories. A relabeling to non-expected welfare or to a Bernoulli welfare function would appear somewhat peculiar.

as the larger system of which humans are but a part" (Raffensperger & Tickner 1999, 353). However, some sort of assessment for the uncertainty is needed. The *minimal information* would be that some harming scenario is deemed possible while others are not. Such a situation is formalized by Arrow & Hurwicz (1972), who show that decision rules coping with that little information have to be based *only* on evaluation of the extreme outcomes in order to satisfy certain rationality properties. With respect to a precautionary evaluation, it seems to be more plausible to base the decision on the evaluation of the worst possible outcome than basing it on the best possible outcome. This reasoning is supported by Bossert, Pattanaik & Xu (2000), who show that in a setting only discriminating between what can and what cannot happen, maximizing the worst possible outcome is the only decision rule conforming with a concept of uncertainty aversion.[17] However, an evaluation that only takes into account the worst possible outcome is usually considered as too extreme. In my example, such a decision criteria implies a willingness to reduce welfare in the first period - in order to avoid the threat of harm - by $u(\bar{x}_2) - \min_{x_2} u(x_2) = u(\bar{x}_2) - u(\underline{x}_2)$. In a scenario where welfare without the threat of harm would coincide in both periods (i.e. $u(x_1) = u(\bar{x}_2)$), this disposition would imply that the decision maker is willing to reduce welfare in the first period to the harm-level $u(\underline{x}_2)$ just to prevent that such welfare level *could* come up in the second period. In my opinion, most decision makers would not subscribe to such an extreme decision rule (including myself). *Therefore, I seek for precautionary evaluation rules*, as a subclass of generalized uncertainty aggregation rules, *that render an evaluation of the uncertain second period, which lies somewhere between expected value and the worst possible outcomes.* These precautionary evaluation rules imply that a decision maker's willingness to reduce welfare in the first period in order to prevent a threat of harm in the second period $p(\underline{x}_2)u(\underline{x}_2)$, is bigger than for the intertemporally additive expected utility maximizer, who uses equation (5.1), i.e. bigger than $u(\bar{x}_2) - \mathrm{E}\, u(x_2)$. This idea of precautionary uncertainty evaluation is formalized in the next section.[18] To this end,

[17]For a definition of what it means precisely to be uncertainty averse in such a setting compare Bossert et al. (2000, Definition 3). The authors also refine the rule of 'maximizing the worst possible outcome' for situations where the extreme outcomes of different scenarios coincide.

[18]In relation to the violation of the independence axiom in the Ellsberg (1961) paradox, the author suggests a decision rule that is a convex combination of the expected utility model and the maximization of the worst possible outcome. This compound decision rule obviously renders an evaluation that lies between expected utility and the valuation of the worst outcome. The latest axiomatization of such a decision rule is given by Chateauneuf, Grant & Eichberger (2003) in the language of subjective Choquet-expected utility by relaxing independence on the extreme outcomes. My axiomatization yields a decision rule that is not a convex combination of these two extremes, but can continuously vary from one extreme to the other. Moreover it stays continuous within the topology of weak convergence.

more information on uncertainty will be needed than the mere specification of what is perceived possible and what is not.

If one is willing to add a little more structure concerning the evaluation of uncertainty than in the Arrow-Hurwicz setup, models of *ambiguity* such as Gilboa & Schmeidler (1989) or its recent generalization by Ghirardato, Maccheroni & Marinacci (2004) are applicable. These models work with sets of probability distributions instead of unique probabilities. In Gilboa & Schmeidler's (1989) axiomatization, decision makers use the worst probability distribution deemed possible to assess an uncertain situation. With respect to the relation between ambiguity and precaution, Gollier (2001, 310 et seq.) criticizes such an attitude as too extreme. However, the recent generalization by Ghirardato et al. (2004) gives a more satisfactory axiomatization, which allows for a much broader and more reasonable class of ambiguity attitudes. In my opinion, their approach resolves Gollier's criticism and supports that, when decision makers are not willing or able to assign a single probability distribution to outcomes, but rather a set of such distributions, there is evidence that some sort of more precautionary decision rule can be needed in order to represent general preferences.

However, I show that it is by no means necessary to abandon the *uniqueness of probabilities* (or the independence axiom, see below), in order to derive the necessity in a preference representation to allow for more precautionary evaluation rules than the one corresponding to equation (5.1). I demonstrate that already the standard von Neumann-Morgenstern assumptions give rise to more precautionary decision rules, as soon as time structure is introduced and taken serious. I consider my study as complementary to other approaches towards extending the notion of uncertainty evaluation, as for example the mentioned study by Ghirardato et al. (2004), which comprises many of the earlier extensions of the von Neumann-Morgenstern setup. Merging my conception of intertemporal risk aversion and these extended treatments of 'atemporal' uncertainty, constitutes a promising research agenda for the future. In the remainder of this section, I briefly discuss the related literature, both, more broadly on choice under uncertainty and more specifically on attempts to formalize the precautionary principle.

Let me start with a closer look at the representation of general uncertainty attitude by Ghirardato et al. (2004). Key to this approach as compared to the standard approach of von Neumann & Morgenstern (1944), and its subjective formulation by Savage (1972), is the relaxation of the independence axiom. The independence axiom roughly states the following.[19] Let a decision maker be indifferent between a lottery p and another lottery p'. Now offer him two compound lotteries, which both start out with a coin toss. In both

[19]A formal statement of the independence axiom is given on page 73.

lotteries the decision maker enters the same third lottery p'' if head comes up. However, if tail comes up, the decision maker faces lottery p in the first compound lottery and the lottery p' in the second. Recalling that the decision maker is indifferent between lotteries p and p', the independence axiom requires the decision maker to be indifferent between the two compound lotteries as well. However, in particular decision contexts, there is ample evidence that observed behavior consistently deviates from the theories of von Neumann & Morgenstern (1944) and Savage (1972), which are built on the independence axiom. The most prominent violations are those discovered by Allais (1953) and Ellsberg (1961). More recent challenges include the equity premium puzzle (see e.g. Kocherlakota 1996) and Rabin's (2000) paradoxical relation between risk aversion in the small and risk aversion in the large for an expected utility maximizer. Note that the first puzzle can partly be explained by disentangling the coefficients of risk aversion and intertemporal substitutability without giving up the independence axiom. Such a disentanglement is also an important step in my formalization of precaution (see chapter 7). Rabin's (2000) paradox has recently been shown to hold for a wide class of non-expected utility theories that give up the independence axiom as well (Safra & Segal 2005). For an overview on other violations of independence compare for example Starmer (2000) and Luce (2000). Despite these behavioral inadequacies, the independence axiom still has a strong normative appeal in the sense of agreeing on a principled approach to evaluate uncertain outcomes as pointed out for example in Hammond (1988*a*), Hammond (1988*b*) and Starmer (2000, 334). For an introduction to approaches for decision-making under uncertainty that abandon or relax the independence axiom, most importantly Quiggin's (1982) Rank-Dependent Utility, Machina's (1982) Local Expected Utility, Choquet Expected Utility dating back to Schmeidler (1989) and Kahneman & Tversky's (1979) Prospect Theory, compare for example Karni & Schmeidler (1991), Schmidt (1998) and Starmer (2000). Note that most of the mentioned theories are completely or to some extent contained in the above mentioned model by Ghirardato et al. (2004).

For the representation of uncertainty by a unique *probability* distribution, there exist different conceptions and axiomatizations. The classical treatments are the frequentist characterization of probability by von Mises (1939) and the measure theoretic axiomatization of Kolmogorov (1933). A quite different approach by de Finetti (1937) derives probabilistic reasoning from assumptions on betting behavior. Savage (1972) recovers probabilistic beliefs and evaluation of outcomes in a joint framework. His framework is also the stepping stone for Ghirardato et al.'s (2004) axiomatization of choice under uncertainty, where relaxing independence goes along with a description of uncertainty in terms of sets of probabilities.[20] Better suited to the context of my study are the epistemic axiomatizations of probability by Koopman (1940) and Cox (1946,1961). Both are partly inspired by the seminal work of Keynes (1921) and construct a probabilistic logic. A more recent treatment within this line of thought is Jaynes (2003). For an overview over the different conceptions of probability compare, for example, Eisenführ & Weber (2003). For a more detailed discussion of subjective probabilities see Kyburg & Smokler (1964) and Fishburn (1986).[21] Note that, as a consequence of sticking to unique probability measures for the description of uncertainty, I will not make a distinction between the words 'risk' and 'uncertainty'.[22]

Complementary attempts on formalizing aspects of the precautionary principle have been carried out by Gollier, Jullien & Treich (2000), Gollier (2001), Gollier & Treich (2003), Immordino (2000), Immordino (2003) and Barrieu & Sinclair-Desgagné (2005). A decade earlier Kimball (1990) analyzed and labeled the concept of precautionary savings. He defined a precautionary savings motive by the condition that an increase of uncertainty over future income raises the current savings. Within an intertemporally additive expected utility model, Kimball (1990) derives conditions on the utility function for the precautionary savings motive to hold. To this end, he defines measures of relative and absolute prudence that characterize the curvature of marginal utility. They are exact analogues to the Arrow-Pratt measures of relative and absolute risk aversion, just applied to marginal utility instead of the utility function itself. Kimball finds that the relation between the measures of risk aversion and prudence determine the precautionary savings motive. Eeckhoudt & Schlesinger (2005) give an axiomatic characterization of prudence and, moreover, extend the concept and the axiomatic characterization to any order of derivatives of the utility function.

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[20] Note however that there is some evidence within the Savage framework that dynamic consistency implies the existence of unique probabilistic beliefs (Epstein & Breton 1993).

[21] For a definition of objective probability beyond the frequency or symmetry definition compare also Popper's (1959) concept of propensity.

[22] A notion frequently found in the literature discussing different forms of uncertainty and ignorance is the following. Risk refers to the particular form of uncertainty where probabilities are known, while general uncertainty also accounts for situations where the probabilities are unknown to the decision maker. This distinction goes back to Knight (1921). Within the concept of epistemic probabilities or subjective probabilistic beliefs, however, it is not obvious what is meant by 'known probabilities'. A possible answer would be to single out objective probabilities as known probabilities. However, among the advocates of subjective probability, there is generally no agreement on the existence of objective probability in the first place. Another distinction possibility is to identify general uncertainty with the concept of ambiguity (or hard uncertainty). In the literature discussed above, the latter corresponds to the non-uniqueness of probabilities, or, in the formulation of Choquet expected utility, to the non-additivity of the (monotonic) set functions (capacities) that replace the concept of probability.

Gollier et al. (2000), Gollier (2001) and Gollier & Treich (2003) analyze a one-commodity, two-period model, in which consumption causes potential damage in the second period. In such a model, they examine the effect of new information that arrives between the first and the second period. The authors label a decision rule precautionary if it satisfies the following condition. Whenever better information[23] about the future is expected, the decision rule must imply a reduction of the (potentially harmful) first period consumption as compared to a situation where no information is expected. They derive a criterion for their concept of precaution to hold in terms of absolute prudence dominating (twice absolute) risk aversion. Gollier (2001, 312) points out that for decision makers exhibiting constant relative risk aversion, this condition is usually regarded unlikely to hold.[24] While the above model considers the reaction of a decision maker in terms of reducing the amount of potential harm by reducing first period consumption, Immordino (2000) and Immordino (2003) explore the decision maker's willingness to invest into a reduction of the probability that the harmful event takes place. Using the terminology of Ehrlich & Becker (1972), Immordino calls actions that reduce the harm level in case it occurs self-insurance and actions that reduce the probability of the potential harm to take place self-protection. In a similar setup as Gollier et al. (2000), Immordino analyzes under which circumstances decision rules exhibit precautionarity in the sense of self-protection. Finally, Barrieu & Sinclair-Desgagné (2005) define a precautionary strategy (within a static setting) as an action that either is self-insuring or self-protecting. They derive a set of mathematical conditions that a precautionary decision maker has to satisfy when confronted by a threat of harm.

While all of the above models, which explicitly refer to the precautionary principle, stay within the intertemporally additive expected utility framework, Kimball & Weil (2003) extend Kimball's (1990) analysis of precautionary savings to a the framework of Kreps & Porteus (1978). They show that in the generalized framework, which allows to distinguish between risk aversion and intertemporal substitutability, all three quan-

[23]This is better information in the sense of Epstein (1980) going back to Marschak & Miyasawa (1968). It can be defined roughly as follows. Let there be two periods and a given expectation for the outcome of the second period at the beginning of the first. After having decided upon first period consumption but before choosing consumption in the second period a signal is received. Whenever an information structure allows to derive (in expectation) a higher welfare gain from such a signal for all reasonable welfare functions than does another, the information structure is said to carry better information.

[24]Note that the model crucially depends on a linearity in the trade-off between second period consumption and second period damage. Moreover, the authors assume an interior solution. However, due to the assumed linearity, this existence assumption is not always met. For the above mentioned scenario of constant relative risk aversion, it can be shown that such an interior solution does not exist.

tities, prudence, risk aversion and intertemporal substitutability, jointly determine the precautionary savings motive. As I work out in chapter 7, my conception of precaution crucially depends on the disentanglement between risk aversion and intertemporal substitutability, but not on prudence. Moreover, it is not tied to a specific model of savings or emission reduction. Motivated as a simple intertemporal reasoning on a decision maker's willingness to undergo preventive action, I present a general characterization in terms of the underlying preferences.

5.3 Uncertainty Aggregation Rules

This section defines the concept of general and precautionary uncertainty aggregation rules. Let X be a connected compact metric space. The elements x of X are called consumption levels or, more general, outcomes. They may contain quantifications in terms of real numbers as well as more abstract characterizations, for example, of current climate or the state of an ecosystem before and after an invasive species has been introduced. The space of all continuous functions from outcomes into the reals is denoted by $\mathcal{C}^0(X)$. More generally, the space of all continuous functions from some metric space Y into the reals is denoted by $\mathcal{C}^0(Y)$. An element $u \in \mathcal{C}^0(X), u : X \to \mathbb{R}$, is called a Bernoulli utility function.[25] Define $\underline{U} = \min_{x \in X} u(x)$, $\overline{U} = \max_{x \in X} u(x)$ and $U = [\underline{U}, \overline{U}]$ so that the range of u is given by U.[26] The set of all Borel probability measures on X is denoted by $P = \Delta(X)$ and equipped with the Prohorov metric which gives rise to the topology of weak convergence. The elements $p \in P$ are called lotteries. Given the epistemic probability definition I referred to in the preceding section, lotteries do not only describe draws from an urn, but are general characterizations of uncertainty with respect to possible outcomes. The degenerate lotteries giving weight 1 to outcome x are denoted by $x \in P$. A lottery yielding outcome x with probability $p(x) = \lambda$ and outcome x' with probability $p(x') = 1 - \lambda$ is written as $\lambda x + (1 - \lambda)x' \in P$. Note that the 'plus' sign between elements of X always characterizes a lottery.[27] Again more generally, the

[25] A more specified definition of Bernoulli utility in relation to the representation of preference relations is given in the next section (compare page 73).

[26] Note that compactness of X and continuity of u assure that the minimum and the maximum are attained.

[27] As X is only assumed to be a compact metric space there is no immediate addition defined for its elements. In case it is additionally equipped with some vector space or field structure, the vector addition will not coincide with the "+" used here. The "+" sign used here alludes to the additivity of probabilities.

set of Borel probability measures on any compact metric space Y is denoted by $\Delta(Y)$. Finally, I denote with $\mathbb{R}_+ = \{z \in \mathbb{R} : z \geq 0\}$ and $\mathbb{R}_{++} = \{z \in \mathbb{R} : z > 0\}$ the set of all positive, respectively strictly positive, real numbers.

An *uncertainty aggregation rule* is defined as a functional $\mathcal{M} : P \times \mathcal{C}^0(X) \to \mathbb{R}$. It takes as input the decision maker's perception of uncertainty, expressed by the probability measure p, and his evaluation of certain outcomes, expressed by his Bernoulli utility function u. For certain outcomes uncertainty aggregation rules are imposed to return the value of the Bernoulli utility itself, i.e. $\mathcal{M}(x, u) = u(x)$. The uncertainty aggregation rule generated by the axiomatization in the subsequent chapter is the following. For a strictly monotonic and continuous function $f : \mathbb{R} \to \mathbb{R}$ define $\mathcal{M}^f : P \times \mathcal{C}^0(X) \to \mathbb{R}$ by

$$\mathcal{M}^f(p, u) = f^{-1}\left[\int_X f \circ u \, dp\right] , \tag{5.2}$$

where $f \circ u$ denotes the usual composition of two functions.[28] The composition sign will often be omitted. This shall not create confusion, as usual multiplication of two functions does not appear within this model. If the probability measure would be defined directly on the range of u, the expression in equation (5.2) would be known as the generalized or f-mean. It aggregates the utility values weighted by some function f and applies the inverse of f to normalize the resulting expression. The only difference between the mean and the uncertainty aggregation rule is that the latter takes the Bernoulli utility function as an explicit argument. If such a correspondence between a mean and an uncertainty aggregation rule holds, I say that the uncertainty aggregation rule (here $\mathcal{M}^f$) is induced by the mean (here generalized or f-mean).[29] The reason for taking up the function u as an explicit argument in the uncertainty aggregation rule, is to explore the freedom in the choice of Bernoulli utility and to stress the similarity between uncertainty aggregation and intertemporal aggregation, which will be introduced in chapter 6.2.

To illustrate the uncertainty aggregation rule $\mathcal{M}^f$ with some examples, let me consider the subset of lotteries having finite support, i.e. the set of all simple probability measures $P^s \subset P$ on X. Then, equation (5.2) can be written as

$$\mathcal{M}^f(p, u) = f^{-1}\left[\sum_x p(x) f \circ u(x)\right] .$$

[28]Note that by continuity of $f \circ u$ and compactness of X Lesbeque's dominated convergence theorem (e.g. Billingsley 1995, 209) ensures integrability.

[29]Precisely this relation can be defined as follows. Let $p^u \in \Delta(U)$ denote the probability measure induced by p defined on X through the Bernoulli utility function $u \in \mathcal{C}^0(X)$ on its (compact) range U. Then an uncertainty aggregation rule $\mathcal{M}$ is said to be induced by a mean $\overline{\mathcal{M}} : \Delta(U) \to \mathbb{R}$ whenever $\mathcal{M}(p, u) = \overline{\mathcal{M}}(p^u) \; \forall p \in P$. Mean inducedness implies that only the probability of x is used to weigh $u(x)$.

The simplest uncertainty aggregation rule corresponds to the expected value operator, and is obtained for $f = \mathrm{id}$:

$$\mathrm{E}(p,u) \equiv \mathrm{E}_p\, u(x) = \sum_x p(x)u(x)\,.$$

It is induced by the arithmetic mean. For Bernoulli utility functions with a range restricted to $U \subseteq \mathbb{R}_+$ another example of an uncertainty aggregation rule is induced by the geometric mean and corresponds to $f = \ln$:

$$\mathrm{G}(p,u) = \prod_x u(x)^{p(x)}\,.$$

Both of the above uncertainty aggregation rules are, again assuming $U \subseteq \mathbb{R}_+$, contained as special cases in the following uncertainty aggregation rule achieved by $f(z) = z^\alpha$:

$$\mathcal{M}^\alpha(p,u) \equiv \mathcal{M}^{\mathrm{id}^\alpha}(p,u) = \left[\sum_x p(x)u(x)^\alpha\right]^{\frac{1}{\alpha}}$$

defined for $\alpha \in \mathbb{R}$ with $\mathcal{M}^0(p,u) \equiv \lim_{\alpha\to 0}\mathcal{M}^\alpha(p,u) = \mathrm{G}(p,u)$ and $\mathcal{M}^1(p,u) = \mathrm{E}(p,u)$.[30] The corresponding mean is known as power mean. In the limit, where α goes to infinity respectively minus infinity, the uncertainty aggregation rule $\mathcal{M}^\alpha$ only considers the extreme outcomes (abandoning continuity in the probabilities): $\mathcal{M}^\infty(p,u) \equiv \lim_{\alpha\to\infty}\mathcal{M}^\alpha(p,u) = \max_x\ u(x)$ and $\mathcal{M}^{-\infty}(p,u) \equiv \lim_{\alpha\to-\infty}\mathcal{M}^\alpha(p,u) = \min_x\ u(x)$.

Let me take $\mathcal{M}^\alpha$ as an example to illustrate the intuition of uncertainty aggregation rules. Assume that an exogenously given u specifies some cardinally measurable welfare information for the outcomes $x \in X$.[31] Now consider a lottery yielding $\bar{\mathfrak{u}} = u(\bar{x}) = 100$ with probability $\bar{p} = 0.9$ and $\underline{\mathfrak{u}} = u(\underline{x}) = 10$ with probability $\underline{p} = 0.1$. Then an expected value maximizer will evaluate the lottery by the certainty equivalent $\mathfrak{u}^c_E = \mathfrak{u}^c_{\alpha=1} = 91$. Another person, who is extremely precautious, might value the lottery only as high as the worst of its outcomes, that is $\mathfrak{u}^c_{\min} = \mathfrak{u}^c_{\alpha=-\infty} = 10$. However, as discussed in the preceding section, the latter is considered as too extreme an assessment. As motivated in the respective setup, a general precautionary decision rule should go along with an uncertainty aggregation rule that renders an evaluation lying somewhere between expected welfare and the welfare of the worst possible outcome. Rewriting the scenario evaluation of equation (5.1) with a general uncertainty aggregation rule yields

[30] The easiest way to recognize the limit for $\alpha \to 0$ is to note that for any $\alpha > 0$ the function $f_\alpha(z) = \frac{z^\alpha - 1}{\alpha}$ is an affine transformation of $f(z) = z^\alpha$. As shown in a ‘Note’ on page 195 in the appendix, affine transformations leave the uncertainty aggregation rule unchanged. Then the fact that $\lim_{\alpha\to 0}\frac{z^\alpha-1}{\alpha} = \ln(z)$ gives the result.

[31] Note that it will be a major task of the subsequent chapters to render a sound basis to this cardinality.

the evaluation rule

$$u(x_1) + \mathcal{M}(p, u)\,. \tag{5.3}$$

The evaluation rule is considered precautionary if it yields a higher willingness to reduce first period welfare in order to avoid a threat of harm than does equation (5.1). Denote by $P_u^{th} = \{p \in P^s : p(\bar{x}_2), p(\underline{x}_2) > 0, p(\bar{x}_2) + p(\underline{x}_2) = 1 \text{ with } \bar{x}_2, \underline{x}_2 \in X, u(\bar{x}_2) > u(\underline{x}_2)\}$ the set of potential *t*hreat of *h*arm lotteries. Then, with an intertemporal evaluation rule of the form (5.3), the definition of a precautionary uncertainty evaluation in the sense of the preceding section can be formalized.

Definition: Given an evaluation of the certain outcomes by a function $u \in \mathcal{C}^0(X)$, an uncertainty aggregation rule is called precautionary if the following relation holds:

$$u(\underline{x}_2) < \mathcal{M}(p, u) < \mathrm{E}(p, u) \quad \forall\, p \in P_u^{th}\,. \tag{5.4}$$

For the uncertainty aggregation rule $\mathcal{M}^\alpha$ it can be shown that the smaller is α, the lower is the certainty equivalent that the respective uncertainty aggregation rule, corresponding to the power mean, brings about (e.g. Hardy, Littlewood & Polya 1964, 26). Hence, within this setup, a precautionary decision maker would be expected to choose a parameter $\alpha < 1$. More generally, in the case of the uncertainty aggregation rule $\mathcal{M}^f$, which is parameterized by the function f, precautionarity of the uncertainty aggregation rule is characterized by the concavity of f as stated in the following proposition.

Proposition 6: An uncertainty aggregation rule of the form $\mathcal{M}^f$ is precautionary in the sense of equation (5.4), if and only if, f is either strictly increasing and concave on U or strictly decreasing and convex on U. An uncertainty aggregation rule of the form $\mathcal{M}^\alpha$ is precautionary in the sense of equation (5.4), if and only if, $\alpha < 1$.

The next chapter develops representations of preferences that make use of uncertainty evaluation by means of $\mathcal{M}^f$. Therefore, I will use the wording precautionary evaluation or uncertainty aggregation rule as a reference to evaluation that goes along with an uncertainty aggregation rule of type $\mathcal{M}^f$ and that satisfies the concavity condition stated in the proposition. Other precautionary uncertainty aggregation rules (like e.g. the one in footnote 18) will be ruled out by the axioms.

Chapter 6

The Representation

6.1 Atemporal Uncertainty

Chapter 6 develops the representational background for the subsequent discussion on intertemporal risk aversion and precaution in chapter 7. This section 6.1 revisits the atemporal von Neumann-Morgenstern setting. Special attention is payed to different possibilities of fixing (gauging) the originally ordinal utility function over the certain outcomes. Section 6.2 takes a brief look at the other framing scenario of additively separable preferences over certain consumption paths, and introduces the concept of an intertemporal aggregation rule. The two framing scenarios are united in section 6.3, where a representation for the simplest setting sufficiently rich to discuss most of the topics pointed out in the introduction is given. The distinctive feature of the representation is that it leaves open the choice of the representing Bernoulli utility function. Section 6.4 points out how fixing (gauging) Bernoulli utility in different ways leads to different representations found in the literature. The idea of keeping some freedom in the choice of Bernoulli utility is already introduced below in the atemporal framework. While it might appear a little artificial at this point, it will prove helpful in later sections and chapters.

A useful perspective on the study in this section is the following. Choice in a certain, atemporal (or one period) setting determines the utility or evaluation function on the certain outcomes only up to strictly increasing transformations. Introducing uncertainty, von Neumann & Morgenstern (1944) single out a particular utility function evaluating the outcomes, by prescribing that expected value maximization should describe choice over lotteries. In other words, they use the originally ordinal character of utility on certain outcomes in order to render a desired uncertainty aggregation rule. If, however, a cardinal evaluation of certain outcomes is given and the freedom of Bernoulli utility no longer prevails, additive representations no longer suffice to represent all decision rules conforming with the von Neumann-Morgenstern axioms. Such a situation can arise, when there is additional information on welfare, for example stemming from intertemporal considerations carried out in the later sections.

For a slightly different perspective on this reasoning, let me introduce a notion borrowed from physics. I call a degree of freedom in a theory, that has no observable effect, a gauge. Fixing this freedom in order to yield a particular representation is called *gauging*. A more familiar wording for gauging is obviously *choosing a normalization*. However, there is a small but important semantic difference between the two concepts. A normalization is usually carried out at the very beginning of an analysis in order to simplify the subsequent algebra. On the other hand, identifying a particular gauge goes along with the reasoning that exploring the freedom of the gauge, instead of eliminating it right away, can render a deeper understanding of a theory. Chapter 7.1 applies this technique to identify the quantity that describes intertemporal risk aversion and precaution. Moreover, carrying along the gauge freedom of Bernoulli utility for a little while, allows to develop different representations which are useful for different questions and interpretations later on in sections 6.4, 7.1 and 7.3.

I represent preferences over lotteries in the usual way by a binary relation on P denoted $\succeq$. For two lotteries $p, p' \in P$ the interpretation of $p \succeq p'$ is that lottery p is weakly preferred with respect to lottery p'. The relation $\succeq$ will be required to be reflexive.[1] The asymmetric part of the relation $\succeq$ is denoted by $\succ$ and interpreted as a strict preference. The symmetric part of the relation $\succeq$ is denoted by $\sim$ and interpreted as indifference. An uncertainty aggregation rule is said to represent the preference relation $\succeq$ over lotteries if

$$p \succeq p' \Leftrightarrow \mathcal{M}(p, u) \geq \mathcal{M}(p', u) \qquad \text{for all } p, p' \in P \tag{6.1}$$

and some $u \in \mathcal{C}^0(X)$. It is said to represent $\succeq$ for $u^* \in \mathcal{C}^0(X)$ if equation (6.1) holds with $u = u^*$. The theorem by von Neumann & Morgenstern (1944), in a version close to Grandmont (1972, 49), states the following.

Theorem 1 (von Neumann-Morgenstern): The axioms

A1 (weak order) $\succeq$ is transitive and complete, i.e.:

[1]Note that reflexivity is implied by completeness in axiom A1.

– transitive: $\forall\, p, p', p'' \in P:\ p \succeq p'$ and $p' \succeq p'' \Rightarrow p \succeq p''$

– complete: $\forall\, p, p' \in P: p \succeq p'$ or $p' \succeq p$

A2 (independence) $\forall\, p, p', p'' \in P:$
$$p \sim p' \quad \Rightarrow \quad \lambda\, p + (1-\lambda)\, p'' \sim \lambda\, p' + (1-\lambda)\, p'' \quad \forall\, \lambda \in [0,1]$$

A3 (continuity) $\forall p \in P: \{p' \in P: p' \succeq\ p\}$ and $\{p' \in P: p \succeq p'\}$ are closed in P

hold, if and only if, there exists a continuous function u: $X \to \mathrm{I\!R}$ such that
$$\forall\, p, p' \in P: \qquad p \succeq p' \;\Leftrightarrow\; \mathrm{E}_p\, u(x) \geq \mathrm{E}_{p'}\, u(x).$$

The theorem states that, accepting axioms A1-A3, there ***exists*** a Bernoulli utility function $\boldsymbol{u}$ on the outcomes such that the uncertainty aggregation rule is of the expected utility form. A1 assumes that the decision maker can rank all lotteries (completeness). Moreover, if one is preferred to a second and the second is preferred to a third, then the first should also be preferred to the third (transitivity). Note that, within the context of deriving a principled approach to choice under uncertainty, A1 should be interpreted as "if a decision maker had the capacities to rank all possible outcomes, then his ranking should satisfy transitivity" rather than as an assumption that the decision maker has actually worked out a ranking of all possible outcomes. The independence axiom A2 has already been discussed in chapter 5.2 on page 62. Continuity A3 assures that infinitesimally small changes in the probabilities do not result in finitely large changes in the evaluation.

Now, consider what happens in a situation, where a decision maker ***has*** a given evaluation $\boldsymbol{u}$ for the certain outcomes.[2] To answer this question, I adapt my earlier definition of Bernoulli utility to the duty of preference representation. The minimal requirement for a utility function to express evaluation of certain outcomes is that a certain outcome x is preferred over a certain outcome x', if and only if, the value assigned to x is higher than that assigned to x'. I call the set of all utility functions, which satisfy this ordinal requirement, the set of *Bernoulli utility functions* $B_{\succeq} = \{u \in \mathcal{C}^0(X) : x \succeq x' \Leftrightarrow u(x) \geq u(x')\, \forall x, x' \in X\}$ for a given preference relation $\succeq$. Note that for convenience of presentation, the definition of Bernoulli utility functions assumes continuity (which is also implied by axiom A3). It is a trivial consequence of theorem 1 that, if the preference relation $\succeq$ satisfies axioms A1-A3, the set of Bernoulli utility functions is nonempty. Moreover, with any Bernoulli utility function $u \in B_{\succeq}$, also a strictly increasing and continuous transformation of u is in $B_{\succeq}$. With regard to my earlier

[2]Wherefrom such a cardinal evaluation may stem will be subject of the subsequent sections. The key to the answer rests within the intertemporal structure.

introduction of a Bernoulli utility function in chapter 5.3 without reference to preference relations, note that any $u \in \mathcal{C}^0(X)$ is a Bernoulli utility function in the sense above for some preference relation $\succeq$. Now, let me specify the uncertainty aggregation rules that represent the decision makers preference over lotteries in the sense of equation (6.1) with a ***given*** Bernoulli utility function $u \in B_{\succeq}$ for a preference relation $\succeq$, satisfying the von Neumann-Morgenstern axioms.

Proposition 7: Given a binary relation $\succeq$ on P and a Bernoulli utility function $u \in B_{\succeq}$ with range U, the relation $\succeq$ satisfies axioms A1-A3, if and only if, there exists a strictly monotonic and continuous function $f : U \to \mathbb{R}$ such that for all $p, p' \in P$

$$p \succeq p' \Leftrightarrow \mathcal{M}^f(p, u) \geq \mathcal{M}^f(p', u).$$

Moreover, if f represents $\succeq$ in the above sense, then f': $U \to \mathbb{R}$ represents $\succeq$ in this sense, if and only if, there exist $a, b \in \mathbb{R}, a \neq 0$ such that $f' = a\, f + b$.[3]

Note that the indeterminacy of f up to affine transformations does not translate into an indeterminacy of the functional $\mathcal{M}$. A function $f' = a\, f + b$ with $a, b \in \mathbb{R}, a \neq 0$ renders the same uncertainty aggregation rule as f, that is $\mathcal{M}^f(\cdot, \cdot) = \mathcal{M}^{f'}(\cdot, \cdot)$. This holds, as the inverse f'^{-1} cancels out the affine displacement of f' with respect to f. In what follows, the group of nondegenerate affine transformations will be denoted by $\mathsf{A} = \{\mathfrak{a} : \mathbb{R} \to \mathbb{R} : \mathfrak{a}(z) = a\, z + b\,,\ a, b \in \mathbb{R}, a \neq 0\}$ with elements $\mathfrak{a} \in \mathsf{A}$, and the group of positive affine transformations will be denoted by $\mathsf{A}^+ = \{\mathfrak{a}^+ : \mathbb{R} \to \mathbb{R} : \mathfrak{a}^+(z) = a\, z + b\,,\ a, b \in \mathbb{R}, a > 0\}$. Then, the uniqueness result of proposition 7 can be written as '...if and only if, there exists $\mathfrak{a} \in \mathsf{A}$ such that $f' = \mathfrak{a} f$.' In later propositions, this notation yields a significant simplification in the formulation of the uniqueness results. In all of the upcoming propositions, corollaries and theorems, the uniqueness result will be stated in a similar form at the end of the proposition (corollary, theorem). Therefore, I will often refer to the uniqueness result as the 'moreover part' of the corresponding proposition.[4]

Let me come back to the perspective given at the beginning of this section. Choice under certainty only renders ordinal information on the Bernoulli utility function u and, thus, can be represented by all members of $B_{\succeq}$. Proposition 7 states that this gauge freedom for Bernoulli utility u translates into the representing uncertainty aggregation

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[3]The theorem can also be stated using only increasing versions of f. In this case $\mathcal{M}^\alpha$ would be included in $\mathcal{M}^f$ in a less obvious way than by $f(z) = z^\alpha$. Strictly decreasing functions are allowed in the proposition, because the inverse in (5.2) cancels out any nondegenerate affine transformation.

[4]Similarly, the proof of the uniqueness results is given in a 'moreover part' subsequent to the proof of the main assertion of the propositions, corollaries and theorems.

rule through the form of the parameterizing function f. Taking this correspondence the other way round one obtains

Corollary 1: For any strictly monotonic, continuous function $f : \mathbb{R} \to \mathbb{R}$ the following assertion holds:
A binary relation $\succeq$ on P satisfies axioms *A1-A3*, if and only if, $\mathcal{M}^f$ represents $\succeq$. The latter is: there exists a continuous function $u : X \to \mathbb{R}$ such that

$$\forall p, p' \in P : \qquad p \succeq p' \Leftrightarrow \mathcal{M}^f(p, u) \geq \mathcal{M}^f(p', u). \tag{6.2}$$

Moreover, if u represents $\succeq$ in the sense of equation (6.2) then u': $X \to \mathbb{R}$ represents $\succeq$ in this sense if and only if there exists $\mathfrak{a}^+ \in \mathsf{A}^+$ such that $u = f^{-1}\mathfrak{a}^+ f\, u'$.[5]

The evaluation function u will obviously be a member of $B_{\succeq}$, as for any f it holds that $\mathcal{M}^f(x, u) = u(x)$. Hence, u represents choice on the certain outcomes in the sense of the definition of a Bernoulli utility function. The uniqueness of u is, in general, no longer up to affine transformations as in theorem 1. Indeterminacy of the Bernoulli utility function u corresponds to those transformations of u, which result in affine transformations of f, and, thus, leave the uncertainty aggregation rule unchanged. For example, in the representation where uncertainty aggregation corresponds to the geometric mean, the remaining gauge freedom of u, after fixing $f = \ln$, is expressed by the group of transformations $u \to c\, u^d$, $c, d \in \mathbb{R}_{++}$.[6]

Corollary 1 points out, how Bernoulli utility functions and uncertainty aggregation rules always come in pairs. For f increasing and strictly concave (or in particular $\mathcal{M}^{\alpha<1}$), corollary 1 reproduces von Neumann-Morgenstern's theorem with expected value replaced by a precautionary uncertainty aggregation rule (compare proposition 6). An immediate consequence of corollary 1 is that

> **In the *atemporal framework* a dispute on whether to apply a precautionary uncertainty aggregation rule or expected value cannot be distinguished from (or can be stated as) a disagreement on the evaluation function over the certain outcomes.**

[5] Recall that $f^{-1}\mathfrak{a}^+ f\, u'$ describes the composed function $f^{-1} \circ \mathfrak{a}^+ \circ f \circ u'$ and not a multiplication of values. Note that equation (6.2) uses f only on the restricted domain U. Alternatively one can define $f : U \to \mathbb{R}$ on a nondegenerate interval U and require $u : X \to U$ to be surjective. Then the representing u in equation (6.2) is unique. Compare to the analysis in chapter 7.4.

[6] Setting $f = \ln$ corresponds to the remaining freedom $u = f^{-1}\mathfrak{a}^+ f\, u' = e^{a \ln(u') + b} = u'^a e^b$ with $a > 0$.

However, the definition of a precautionary uncertainty aggregation rule in chapter 5.3 appealed to an intertemporal setup, where the evaluation of uncertainty in the second period determined the willingness to undergo prevention effort in the first period. Such a time structure proves to be essential to define the concept of precaution in terms of preferences. As a consequence, temporality is introduced in the next section.

6.2 Intertemporal Certainty

This short section treats the other framing scenario of the general framework, i.e. additively separable preference over certain outcome paths. Time is discrete with planning horizon $T \in \mathbb{N}$. Individual periods are usually denoted with time indices $t, \tau \in \{1, ..., T\}$. The set of all certain consumption paths from period t to period T is denoted by $\mathsf{X}^t = X^{T-t+1}$, where X^{T-t+1} denotes the $T-t+1-$fold Cartesian product of X with itself.[7] A consumption path is generally written with a calligraphic x and its period τ entry is denoted by x_τ. When explicit reference is made to the fact that x is an element of X^t I give the consumption path an upper case time index t. Then x^t denotes a consumption path from period t to period T, and x^t_τ denotes the period τ entry of the respective consumption path. For example, a (planned) consumption path $\mathsf{x} \in \mathsf{X}^1$ in period one writes as $\mathsf{x} = (\mathsf{x}_1, \mathsf{x}_2, ..., \mathsf{x}_T)$. Whenever such a notation is unambiguous, I also label the entry x_t by x_t, yielding the notation $\mathsf{x} = (x_1, x_2, ..., x_T)$ for the consumption path. Furthermore, let $x^0 \in X$ be some benchmark consumption. It is arbitrarily fixed and serves to define the shorthand notation $[x]_t \equiv (x, x^0, ..., x^0)$ for the consumption path $[x]_t \in \mathsf{X}^t$ that yields the specified consumption x in period t and the benchmark consumption in all subsequent periods. The benchmark consumption is a common outcome when comparing two different paths of type $[x]_t$ and $[x']_t$. Thus, the relation $[x]_t \succeq_t [x']_t$ expresses a preference of x over x' in period t.[8] The final piece of notation concerns the introduction of an intertemporal aggregation rule. Like the uncertainty aggregation rules in the preceding sections have been parameterized by a function f, the intertemporal aggregation rules will be parameterized by a function g. For a given function $u \in \mathcal{C}^0(X)$ with range U, I define $\underline{G} = g(\underline{U})$, $\overline{G} = g(\overline{U})$ and $G = [\underline{G}, \overline{G}]$. In addition, I define $\Gamma = (\underline{G}, \overline{G})$.

[7] There are $T - t + 1$ periods from t to T for which consumption has to be specified. I do not distinguish different sets of outcomes for different periods. X can be thought of as the union of all possible outcomes perceivable in any period.

[8] The intertemporal separability implied by axiom A4 will make this expression of preference independent of the choice of the benchmark consumption x^0.

In this section, the binary relation $\succeq$ depicts preferences on the set X^1 of certain consumption paths starting in the first period. As mentioned before, I want the model to be additively separable over time with respect to certain outcomes. The reason is that it is the predominant framework for intertemporal modeling, in particular in environmental economics, and eases the economic interpretation (compare the discussion in chapter 7.3). Moreover, despite its simplicity, the additively separable structure on certain outcomes over time proves to be sufficiently rich to analyze the concept of precaution and intertemporal risk aversion. Examples for axiomatizations of additive separability include Koopmans (1960) and Radner (1982). However, these axiomatizations are not within the focus of my analysis. In consequence, I take additive separability as a direct assumption expressed in the following axiom.

A4 (certainty additivity) There exists $u \in \mathcal{C}^0(X)$ such that for all $\mathsf{x}, \mathsf{x}' \in X^1$

$$\mathsf{x} \succeq \mathsf{x}' \Leftrightarrow \sum_{\tau=1}^{T} u(x_\tau) \geq \sum_{\tau=1}^{T} u(x'_\tau). \tag{6.3}$$

Note that this axiom also includes the assumptions of stationarity and a zero rate of time preference. Stationarity implies that the mere passage of time does not have an (anticipated) effect on preferences. For example, if stationarity holds, I will not (anticipate in my plans to) prefer Beck's beer over Budweiser in 2010 and Budweiser over Beck's beer in 2011. The assumption that the pure rate of time preference is zero implies that future well-being is given the same weight as present well-being. Both assumptions will be relaxed in chapter 8. However, for the time being, these assumptions are helpful, as they do not affect qualitatively the concepts of precaution and intertemporal risk aversion, and allow to focus on the essential. In chapters 9 and 10 I derive axiomatizations that imply stationarity and a zero rate of time preference, taking general non-stationary preference as a starting point. Another assumption that is implied by axiom A4, is history independence, which excludes (anticipated) habit formation. An extension of the concept of intertemporal risk aversion to a framework allowing for history dependence of preferences, is not pursued in this study, but constitutes an interesting challenge for future research.

Like in the preceding section, the evaluation functions representing preferred choice on certain one-period outcomes are called Bernoulli utility functions. The respective set $B_\succeq$ characterizing the Bernoulli utility functions is defined by the straight-forward extension $B_\succeq = \{u \in \mathcal{C}^0(X) : [x]_1 \succeq [x']_1 \Leftrightarrow u(x) \geq u(x') \,\forall\, x, x' \in X\}$, which coincides with the definition given in section 6.1 for $T = 1$. Note that axiom A4 ensures that the definition of $B_\succeq$ does not depend on the choice of the benchmark consumption x^0.[9]

[9] Nor does it depend on the fact that the defining paths $[\cdot]$ have constant future consumption streams.

It is important to realize that axiom A4 does not imply that equation (6.3) holds for every Bernoulli utility function $u \in B_{\succeq}$. It only implies that there *exists* a *particular* Bernoulli utility function such that comparisons between different consumption paths can be expressed as comparisons of the sum over per period utility.[10] For an arbitrary **given** Bernoulli utility function $\boldsymbol{u}$ the following analogy to proposition 7 holds.

Proposition 8: Given a preference relation $\succeq$ on X^1 and a Bernoulli utility function $u \in B_{\succeq}$ with range U, the relation $\succeq$ satisfies axiom A4 if and only if there exists a strictly monotonic, continuous function $g : U \to \mathbb{R}$ such that for all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^1$

$$\mathsf{x} \succeq \mathsf{x}' \Leftrightarrow g^{-1}\left[\frac{1}{T}\sum_{t=1}^{T} g \circ u(x_t)\right] \geq g^{-1}\left[\frac{1}{T}\sum_{t=1}^{T} g \circ u(x'_t)\right] . \tag{6.4}$$

Moreover, if $T \geq 2$ and g represents $\succeq$ in the sense of equation (6.4), then g': $U \to \mathbb{R}$ represents $\succeq$ in this sense, if and only if, there exists $\mathfrak{a} \in \mathsf{A}$ such that $g' = \mathfrak{a}g$.

For the case of two periods let me abbreviate the representational form by introducing the notion of an *intertemporal aggregation rule* $\mathcal{N}^g : U \times U \to \mathbb{R}$, $\mathcal{N}^g(\cdot,\cdot) = g^{-1}\left[\frac{1}{2}\, g(\cdot) + \frac{1}{2}\, g(\cdot)\right]$. Such an intertemporal aggregation rule resembles closely the functional form of the uncertainty aggregation rule $\mathcal{M}^f$. Here, the probability weights entering $\mathcal{M}^f(p,u)$ correspond to the period weights $\frac{1}{2}$ and assure that the expression is well defined.[11] The fact that the period weights are fixed, allows to define the intertemporal aggregation rule directly on the range U of the Bernoulli utility function. For the particular Bernoulli utility function corresponding to u in equation (6.3) of axiom A4, the function g is the identity and intertemporal aggregation becomes linear.

[10]Referring to axiom A4 at more length as 'additive separability over certain consumption paths' might be more helpful in making this point. Naming the axiom simply 'certainty additivity' follows Chew & Epstein (1990, 61).

[11]That is that the range of $\frac{1}{2}\, g(\cdot) + \frac{1}{2}\, g(\cdot)$ coincides with the domain of g^{-1}. For a stationary setting with positive discounting, these weights change to the form given in chapter 9.

6.3 Certain × Uncertain

Combining the thoughts of sections 6.1 and 6.2, I now combine time structure and uncertainty. For this end I consider the simplest nontrivial setting: a two period framework, where consumption in the first period is certain and consumption in the second period is uncertain. It is the simplest framework that allows to shed light on the idea of precaution, as it has been motivated in chapter 5.2. I consider such a simplified framework useful to familiarize with the structure of the representation, the idea of gauging and for introducing the concept of intertemporal risk aversion. As I work out in chapter 8, all the essential insights gained in this simplified framework extend in a straight forward way to any finite time horizon. For such a multiperiod extension, I apply the framework of Kreps & Porteus (1978). In their terminology, the time frame discussed here only spans one and a half periods. Anticipating the more general setup and avoiding notational confusion I denote the first period in this section by $t = F$ and the second and last period by $t = T$.[12] Let me finally remark that this short section only presents the mathematical structure. Its economic content will be discussed in chapter 7.

Elements $x_F \in X$ denote certain consumption in the first period. Degenerate lotteries yielding certain consumption in the second period are denoted by $x_T \in X$. If the period, in which outcome x takes place is obvious, the time index is omitted. General objects of choice in the second period are the lotteries $p \in P$, just as in section 6.1. The preference relation over these objects is denoted by $\succeq_T$. Objects of choice in the first period are combinations of certain consumption in the first period and lotteries faced in the next: $(x, p) \in X \times P$. Preferences over these objects are given by the relation $\succeq_F$. The set of preferences in both periods will be denoted by $\succeq = (\succeq_F, \succeq_T)$.

I demand that preferences restricted to certain consumption paths satisfy certainty additivity A4, and that lotteries are evaluated on basis of the von Neumann-Morgenstern axioms A1-A3. In addition, the preferences in period one and two shall be connected by the following consistency axiom:

A5 (time consistency) For all $x \in X$ and $p, p' \in P$: $\quad (x, p) \succeq_F (x, p') \Leftrightarrow p \succeq_T p'$.

This is *time consistency* in the sense of Kreps & Porteus (1978).[13] It is a requirement for choosing between two consumption plans that coincide in their first period outcome. For

[12] Due to backward recursion in the derivation of the general representation, the structure of the second period in this representation corresponds to the last period in the multiperiod setting. A full time-step back would also introduce uncertainty for the preceding (i.e. first) period.

[13] Adapted to the one and a half period setting of this section.

these choice situations, axiom A5 demands that in the first period the decision maker shall prefer the plan that gives rise to the lottery that is preferred in the second period.

Again, I am interested in finding a representation for $\succeq$, for a given evaluation $u \in B_{\succeq} \equiv B_{\succeq_F}$ on the certain one-period outcomes. The definition of the set of Bernoulli functions given in the preceding section still applies. Setting $u \in B_{\succeq} \equiv B_{\succeq_F}$ is justified by the fact that certainty additivity A4 and time consistency A5 imply that $B_{\succeq_F} = B_{\succeq_T}$.[14] Denote by $\succeq_F|_{X\times X}$ the restriction of $\succeq_F$ to the set of certain consumption paths (or pairs). The following representation theorem holds.

Theorem 2: Given a set of binary relations $\succeq = (\succeq_F, \succeq_T)$ on $(X \times P, P)$ and a Bernoulli utility function $u \in B_{\succeq}$ with range U, the set of relations $\succeq$ satisfies

i) A1-A3 for $\succeq_T$ (vNM setting)

ii) A4 for $\succeq_F|_{X\times X}$ (certainty additivity)

iii) A5 (time consistency)

if and only if, there exist strictly monotonic and continuous functions $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ such that

$$v) \quad (x,p) \succeq_F (x',p') \quad \Leftrightarrow \quad \mathcal{N}^g\left[u(x)\,,\, \mathcal{M}^f(p,u)\right] \;\geq\; \mathcal{N}^g\left[u(x')\,,\, \mathcal{M}^f(p',u)\right]$$

$$vi) \quad p \succeq_T p' \quad \Leftrightarrow \quad \mathcal{M}^f(p,u) \;\geq\; \mathcal{M}^f(p',u)\,.$$

Moreover, g and f are unique up to nondegenerate affine transformations.

In this representation, period T lotteries are evaluated the same way as in proposition 7 by means of the uncertainty aggregation rule $\mathcal{M}^f$. In the first period, these second period lottery-evaluations are aggregated with the evaluation of the certain outcomes x_F by means of the *intertemporal aggregation rule* $\mathcal{N}^g$ the same way as in proposition 8. With respect to the roman numbering in the above and later theorems, I adopt the convention that numbers $i - iv$ are related to assumptions on preferences, while numbers starting from v are concerned with the functional representation.[15] Now, it will be interesting to look again at the gauge-freedom within the representation.

[14] This is shown in the proof of theorem 2. Observe that in the definition of $B_{\succeq_F}$ it is $[x] = (x, x^0)$ where x^0 is identified with the degenerate lottery yielding the benchmark outcome x^0 in the second period.

[15] The only exception to this rule will be the very last theorem in this dissertation.

6.4 Gauging

Like in section 6.1, there is some gauge freedom rendered to the model by the freedom to choose the Bernoulli utility function in theorem 2. Given some $u \in B_{\succeq}$, any other Bernoulli utility function is a strictly increasing continuous transformation of u and any strictly increasing continuous transformation of u yields an element of $B_{\succeq}$. Moreover the following lemma holds.

Lemma 1: If the triple (u, f, g) represents the set of preferences $\succeq$ in the sense of theorem 2, then so does the triple $(s \circ u, f \circ s^{-1}, g \circ s^{-1})$ for any $s : U \to \mathbb{R}$ strictly increasing and continuous.

Now, like in section 6.1, I can gauge the uncertainty aggregation rule in the representation of theorem 2 to any desired form which is parameterized by a strictly monotonic and continuous f^*. This is achieved by choosing $s = f^{*-1} \circ f$ in lemma 1, and yields the following corollary of theorem 2.

Corollary 2 (**f-gauge**) :

For any strictly monotonic and continuous function $\boldsymbol{f}$: $\mathbb{R} \to \mathbb{R}$, the following equivalence holds:

A set of binary relations $\succeq$ satisfies

$i - iii$) of theorem 2,

if and only if, there exists a continuous function $u : X \to \mathbb{R}$ with range U and a strictly monotonic and continuous function $g : U \to \mathbb{R}$ such that

$v - vi$) of theorem 2 hold.

Moreover, the pairs (u, g) and (u', g') both represent $\succeq$ in the above sense, if and only if, there exist $\mathfrak{a} \in \mathsf{A}$ and $\mathfrak{a}^+ \in \mathsf{A}^+$ such that the relation $(u, g) = (f^{-1}\mathfrak{a}^+ f\ u', \mathfrak{a}\ g'\ f^{-1}\mathfrak{a}^{+-1} f)$ holds.

The gauge used implicitly by Kreps & Porteus (1978) is obtained for f = id. The latter implies that the uncertainty aggregation rule becomes additive, i.e. expected utility. Then equations v) and vi), characterizing the representation (compare theorem 2), become:

Kreps Porteus gauge *(f = id$-$gauge)* :

$$v)\quad (x,p) \succeq_F (x',p') \quad \Leftrightarrow \quad \mathcal{N}^g\left[u(x)\,,\, E_p\, u\right] \geq \mathcal{N}^g\left[u(x')\,,\, E_{p'}\, u\right]$$

$$vi)\quad p \succeq_T p' \quad \Leftrightarrow \quad \mathrm{E}_p\, u \geq \mathrm{E}_{p'}\, u\ .$$

While uncertainty aggregation is linear in this gauge, aggregating Bernoulli utility of the first periods with expected Bernoulli utility of the second period is, in general, nonlinear. The interpretation of the linearity of uncertainty aggregation at the cost of nonlinearity over time will be discussed in the next chapter. Let me remark that Kreps & Porteus (1978) get a slightly more general intertemporal aggregation rule, for they do not demand certainty additivity in the sense of axiom A4. In the notion of Johnsen & Donaldson (1985), my axiom implies unconditional strong independence over time for certain outcomes while the analogue in their setting would be conditional strong independence, which is slightly weaker. However, axiom A4 allows for a special gauge that will prove most helpful for discussing the meaning of welfare and precaution in chapter 7.3. This gauge is a special case of the following

Corollary 3 (**g-gauge**) :

For any strictly monotonic and continuous function $\boldsymbol{g}$: $\mathbb{R} \rightarrow \mathbb{R}$, the following equivalence holds:

A set of binary relations $\succeq$ satisfies

$i - iii)$ of theorem 2,

if and only if, there exists a continuous function $u : X \rightarrow \mathbb{R}$ with range U and a strictly monotonic and continuous function $f : U \rightarrow \mathbb{R}$ such that

$v - vi)$ of theorem 2 hold.

Moreover, the pairs (u, f) and (u', f') both represent $\succeq$ in the above sense, if and only if, there exist $\mathfrak{a} \in \mathsf{A}$ and $\mathfrak{a}^+ \in \mathsf{A}^+$ such that the relation $(u, f) = (g^{-1}\mathfrak{a}^+ g\, u', \mathfrak{a}\, f'\, g^{-1}{\mathfrak{a}^+}^{-1} g)$ holds.

It renders the above mentioned *certainty additive gauge* for $g = \text{id}$. Setting the function g to identity implies that intertemporal aggregation of Bernoulli utility becomes linear. Then the representation mimics the setting discussed in chapter 5.

Certainty additive gauge *($g = \text{id}-$gauge)* :

$$v) \quad (x,p) \succeq_F (x',p') \quad \Leftrightarrow \quad u(x) + \mathcal{M}^f(p,u) \ \geq \ u(x') + \mathcal{M}^f(p',u),$$

$$vi) \quad p \succeq_T p' \quad \Leftrightarrow \quad \mathcal{M}^f(p,u) \ \geq \ \mathcal{M}^f(p',u).$$

In this gauge, uncertainty aggregation will generally be nonlinear and, thus, differ from taking the expected value. Let me point out that in the general multiperiod framework as introduced in chapter 8.1, the intertemporal aggregation rule $\mathcal{N}^g$ is applied recursively. Due to the nonlinearity in the uncertainty aggregation rules and the recursive evaluation of lotteries, complete additive separability in the sense of an immediate summation over periods is obtained only for certain consumption paths.[16]

Another special gauge is possible, if the outcome space is one-dimensional, i.e. $X \subset \mathbb{R}$, and Bernoulli utility is strictly increasing in the consumption level $x \in X$. Then, the representing Bernoulli utility function u in theorem 2 can be chosen as the identity, rendering immediately the

Epstein Zin gauge *($u = \text{id}-$gauge, one commodity) :*

$$\begin{array}{lccccccc} v) & (x,p) & \succeq_F & (x',p') & \Leftrightarrow & \mathcal{N}^g\left[x\,,\,\mathcal{M}^f p\right] & \geq & \mathcal{N}^g\left[x'\,,\,\mathcal{M}^f p'\right] \\ vi) & p & \succeq_T & p' & \Leftrightarrow & \mathcal{M}^f p & \geq & \mathcal{M}^f p' \end{array}$$

with $\mathcal{M}^f p \equiv \mathcal{M}^f(p, \text{id}) = f^{-1}\left[\int_X f(x)\, dp\right]$.

In this representation, Bernoulli utility is not explicit anymore. Such a representation is used by Epstein & Zin (1989) to distinguish between risk aversion and intertemporal substitutability, as will be discussed in chapter 7.1. The representation assumed by these authors slightly differs from the one supported by my axiomatization. With respect to the intertemporal aggregation rule, Epstein & Zin (1989) assume the special case where $g(z) = z^\rho$, which renders an intertemporal aggregation with a constant elasticity of intertemporal substitution. On the other hand, they assume a more general uncertainty aggregation rule, which does not comply with von Neumann & Morgenstern's (1944) independence axiom. For the one and a half period model discussed here, already Selden (1978) introduced the representation corresponding to the $u = \text{id}$-gauge. However, only Epstein & Zin (1989) give a time consistent multiperiod extension that has been taken up by many authors in order to disentangle risk aversion from intertemporal substitutability. Such a disentanglement is the topic of the next section.

[16]For the evaluation of the uncertain future, only intertemporal aggregation in each recursive step is additive in the certainty additive gauge.

Chapter 7

Discussion

7.1 Risk Aversion and Intertemporal Substitutability

This chapter elaborates the economic interpretation of the representation developed in the preceding chapter and introduces the concept of intertemporal risk aversion. First, this section takes a close look at the disentanglement of intertemporal substitutability and (standard) risk aversion in the one and in the multi-commodity setting. While the latter quantities are seen to be good-dependent, an invariant quantity is identified. Section 7.2 develops an interpretation of this quantity by axiomatically introducing the concept of intertemporal risk aversion. Section 7.3 relates the concept to the discussion on precaution in chapter 5. Section 7.4 elaborates quantitative measures of intertemporal risk aversion and identifies the conditions for uniqueness. Finally, section 7.5 gives a brief summary of the analysis in part II.

It is well known that risk aversion and intertemporal substitutability cannot be distinguished within the standard framework of intertemporally additive expected utility (Weil 1990). For the latter preference specification the intertemporal elasticity of substitution is confined to the inverse of the coefficient of relative risk aversion. However, Epstein & Zin (1989) work out how these two characteristics of preference can be disentangled in the more general setting of Kreps & Porteus (1978). To this end they use a one commodity setting and the Epstein Zin gauge of chapter 6.4:

$\succeq_F$ representation in *Epstein Zin gauge ($u = \text{id}-$gauge)*:

$$\mathcal{N}^g \left[x \, , \, \mathcal{M}^f p \right]$$

with $\mathcal{M}^f p = \mathcal{M}^f(p, \mathrm{id}) = f^{-1}\left[\int_X f(x)\, dp\right]$.

With respect to the intertemporal aggregation rule, Epstein & Zin (1989) assume the special case where $g(z) = z^\rho$, rendering a CES function for intertemporal aggregation. Then, $\sigma = \frac{1}{1-\rho}$ characterizes the elasticity of intertemporal substitution on certain consumption paths, and the authors identify ρ as the parameter reflecting intertemporal substitutability (Epstein & Zin 1989, 949). The easiest way to recognize that the uncertainty aggregation rule characterizes risk attitude,[1] is by comparing two risky second period choices going along with the same first period consumption:

$$\begin{aligned} (x, p) \quad &\succeq_F \quad (x, p') \\ \Leftrightarrow \mathcal{N}^g\left[x\,,\, \mathcal{M}^f p\right] \quad &\geq \quad \mathcal{N}^g\left[x\,,\, \mathcal{M}^f p'\right] \\ \Leftrightarrow \quad \mathcal{M}^f p \quad &\geq \quad \mathcal{M}^f p' \\ \Leftrightarrow \quad \int f(x)\, dp \quad &\geq \quad \int f(x)\, dp' \end{aligned} \tag{7.1}$$

It is well known from the atemporal theory of risk aversion that, for a decision maker whose evaluation of lotteries relies on equation (7.1), the concavity of f plays the essential role in characterizing his risk aversion.[2] For a twice differentiable function f, equation (7.1) reveals the Arrow-Pratt-measure of relative risk aversion as $\mathrm{RRA}(x) = -\frac{f''(x)}{f'(x)}\, x$. The advantage of the Arrow-Pratt-measure of relative risk aversion as opposed to f itself, is that it eliminates the affine indeterminacy of f that prevails by the moreover part of theorem 2. In particular, for $f(z) = z^\alpha$ the coefficient of relative risk aversion becomes $\mathrm{RRA} = -\frac{f''(x)}{f'(x)} x = 1 - \alpha$ (constant relative risk aversion). I adopt the wording that f in the general setting, and α in the particular case of constant relative risk aversion, parametrize uncertainty aggregation and risk attitude. More precisely, they characterize 'a-temporal' risk attitude, which means that in equation (7.1) time plays no role whatsoever. The emphasis of 'a-temporal' is borrowed from Normandin & St-Amour (1998, 268), who use it to point out the difference between the 'inter-temporal' information contained in the parametrization of intertemporal substitutability, and the 'a-temporal' nature of the risk attitude captured by RRA. I come

[1] Note that Epstein & Zin (1989) assume a more general uncertainty aggregation rule, which, in general, does not comply with von Neumann & Morgenstern's (1944) independence axiom.

[2] In the atemporal theory developed by von Neumann & Morgenstern (1944) and stated in theorem 1, f is usually denoted as u and given the interpretation of Bernoulli utility itself. To understand the relation to Bernoulli utility in the intertemporal setting, note first that in the atemporal setting g can be neglected. Then, by lemma 1, the preference representation over the second period lotteries (id, f) corresponding to equation (7.1), is equivalent to the representation $(f \circ \mathrm{id}, f \circ f^{-1}) = (f, \mathrm{id})$. The latter is a representation in the sense of theorem 1. In words, f in the Epstein Zin ($u = \mathrm{id}$-)gauge corresponds to Bernoulli utility u in the classical von Neumann & Morgenstern (1944) theorem.

back to this idea in the next section, when introducing the concept of intertemporal risk aversion. The special case, exhibiting constant elasticity of substitution ($g(z) = z^\rho$) and constant relative risk aversion ($f(z) = z^\alpha$), is also known as generalized isoelastic preference (Weil 1990). Independently of Epstein & Zin (1989, 1991), it has also been developed by Weil (1990). Currently, it represents the predominantly employed model for disentangling risk aversion from intertemporal substitutability. For this special case, the preference representation in the first period writes as follows.

$\succeq_F$ representation in *Epstein Zin gauge ($u =$ id$-$gauge)*, isoelastic case:

$$\left\{\tfrac{1}{2}\, x^\rho + \tfrac{1}{2}\, [\mathcal{M}^\alpha p]^\rho\right\}^{\frac{1}{\rho}}$$

with $\mathcal{M}^\alpha p = \mathcal{M}^\alpha(p, \text{id}) = \left[\int_X x^\alpha \, dp\right]^{\frac{1}{\alpha}}$.

This form has been used in many applications ranging from asset pricing (Attanasio & Weber 1989, Svensson 1989, Epstein & Zin 1991, Normandin & St-Amour 1998, Epaulard & Pommeret 2001) over measuring the welfare cost of volatility (Obstfeld 1994, Epaulard & Pommeret 2003*b*) to resource management[3] (Knapp & Olson 1996, Epaulard & Pommeret 2003*a*, Howitt et al. 2005) and evaluation of global warming scenarios (Ha-Duong & Treich 2004). An overview over the empirical findings for the parameters α and ρ can be found in Giuliano & Turnovsky (2003). Note that the papers mentioned above employ the multiperiod extension of the model with a generally positive discount rate, as introduced in chapters 8 and 9.[4]

The analysis in chapter 6.4 shows that the Epstein Zin gauge is a particular representation for a one commodity setting. By choosing the Bernoulli utility function as the identity, it uses the natural scale of the single consumption commodity to measure risk aversion and intertemporal substitutability. The representing triple in the sense of theorem 2 can be written as (id_X, f, g), where id_X denotes the identity on $X \subset \mathbb{R}$. In this paragraph I work out, how a change in the measure-scale of the commodity generally alters the parameterizations of risk attitude and intertemporal substitutability. In the case of one commodity, the analysis highlights an aspect of gauge-dependence that becomes crucial in the multi-commodity setting. Let $x \in [\underline{X}, \overline{X}] \subset \mathbb{R}_+$ denote the quantity

[3]While Knapp & Olson (1996) and Epaulard & Pommeret (2003*a*) solve theoretical models in order to obtain optimal rules for resource use, Howitt, Msangi, Reynaud & Knapp (2005) try to rationalize observed reservoir management in California, which cannot be explained by means of intertemporally additive expected utility.

[4]I say generally positive, because some estimates, when disentangling α and ρ, actually find a negative discount rate (e.g. Epstein & Zin 1991). For positive discounting the above representation on certain $\times$ uncertain outcomes becomes $\left\{\frac{1}{1+\beta}\, u(x)^\rho + \frac{\beta}{1+\beta}\, \left[\mathcal{M}^\alpha p\right]^\rho\right\}^{\frac{1}{\rho}}$ (see chapter 9). Moreover Svensson (1989) translates the isoelastic model to continuous time, which is also used in Epaulard & Pommeret (2003*b*) and Epaulard & Pommeret (2003*a*).

of the (single) consumption commodity in the original measure-scale. For some increasing continuous transformation $s \in \mathcal{C}^0([\underline{X}, \overline{X}])$, let $\tilde{x} = s(x) \in \tilde{X} = [s(\underline{X}), s(\overline{X})] \subset \mathbb{R}$ denote the quantity of the consumption commodity in the new measure-scale. For example, a change in measurement from kg to g would correspond to $s(x) = 1000\,x$. As u is the identity in the Epstein Zin gauge, such a change of measure-scale can be identified with $u = \mathrm{id}_X \Rightarrow \tilde{u} = s \circ \mathrm{id}_X$ corresponding to $\mathrm{id}_{\tilde{X}}$ in the new measure scale. By lemma 1 it is known that with the triple (id_X, f, g) also the triple $(s \circ \mathrm{id}_X, f \circ s^{-1}, g \circ s^{-1})$ represents $\succeq_F$. But in terms of the new measure scale, the latter writes as $(\mathrm{id}_{\tilde{X}}, f \circ s^{-1}, g \circ s^{-1}) = (\mathrm{id}_{\tilde{X}}, \tilde{f}, \tilde{g})$ with $\tilde{f} = f \circ s^{-1}$ and $\tilde{g} = g \circ s^{-1}$ defined on $\tilde{X}$. These functions $\tilde{f}$ and $\tilde{g}$ are the new parameterizations of risk aversion and intertemporal substitutability for the changed measure-scale. A twice differentiable $\tilde{f}$ is associated with the new coefficient of relative risk aversion

$$\tilde{\mathrm{RRA}}(\tilde{x}) = -\frac{\tilde{f}''(\tilde{x})}{\tilde{f}'(\tilde{x})}\,\tilde{x} = -\frac{1}{s'(x)}\left[\frac{f''\left(s^{-1}(\tilde{x})\right)}{f'\left(s^{-1}(\tilde{x})\right)} - \frac{s''\left(s^{-1}(\tilde{x})\right)}{s'\left(s^{-1}(\tilde{x})\right)}\right]\tilde{x}\,.$$

Comparing relative risk aversion at the same physical consumption level $\tilde{x} = s(x)$ yields the index

$$\tilde{\mathrm{RRA}}(\tilde{x})\Big|_{\tilde{x}=s(x)} = -\frac{\tilde{f}''(\tilde{x})}{\tilde{f}'(\tilde{x})}\,\tilde{x}\Bigg|_{\tilde{x}=s(x)} = -\frac{s(x)}{s'(x)}\left[\frac{f''(x)}{f'(x)} - \frac{s''(x)}{s'(x)}\right]$$

for the new measure-scale as compared to

$$\mathrm{RRA}(x) = -\frac{f''(x)}{f'(x)}\,x$$

for the old measure-scale. Hence, *in general, a change of the measure-scale of the consumption commodity changes the coefficient of relative risk aversion.*[5] However, it is interesting to note that a multiplicative rescaling of the consumption unit leaves the coefficient of relative risk aversion unchanged.[6] Let $s(x) = a\,x$ as in the example of changing the measure from kg to g ($a = 1000$). Then it is

$$\tilde{\mathrm{RRA}}(\tilde{x})\Big|_{\tilde{x}=ax} = -\frac{\tilde{f}''(\tilde{x})}{\tilde{f}'(\tilde{x})}\,\tilde{x}\Bigg|_{\tilde{x}=ax} = -\frac{ax}{a}\left[\frac{f''(x)}{f'(x)} - \frac{0}{a}\right] = -\frac{f''(x)}{f'(x)}x = \mathrm{RRA}(x)\,.$$

[5]Note that this is despite the fact that the coefficient of relative risk aversion is defined in a way to cancel out the unit of the x measurement.

[6]This fact is particularly interesting because indeterminacy up to a multiplicative constant is a frequently encountered form of indeterminacy when defining a measure scale, e.g. for the quantity of an arbitrarily divisible good. In such a situation, the meaning of a zero consumption level is naturally given, and the concept of "double as much" as well. However, the unit has to be fixed by convention, e.g. in the mentioned example to grams. A different fixing of the unit (e.g. to kg or pound) corresponds to a multiplicative rescaling of the measure-scale.

This finding implies that *the coefficients α and ρ in the isoelastic setting do not depend on a scaling factor of the measure units.* Note, however, that a general affine transformation $\mathfrak{a} \in \mathsf{A}^+$ does change the coefficient of relative risk aversion:

$$\tilde{\text{RRA}}(\tilde{x})\Big|_{\tilde{x}=ax+b} = -\frac{\tilde{f}''(\tilde{x})}{\tilde{f}'(\tilde{x})}\,\tilde{x}\Bigg|_{\tilde{x}=ax+b} = -\frac{f''(x)}{f'(x)}\frac{ax+b}{a} \neq \text{RRA}(x)\,.$$

The preceding reasoning on the change of risk measure under a change of measure-scale is intimately linked to the question on how to extend the notion of risk aversion and the disentanglement of risk aversion and intertemporal substitutability to a multi-commodity setting. Let me first explore an example, where a decision maker has preferences over two different types of consumption, x_1 and x_2, quantified on some closed subsets of $\mathbb{R}_+$. Let me assume that his preferences are representable by a Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\gamma_1} x_2^{\gamma_2}$ with $\gamma_1, \gamma_2 > 0$, and furthermore that $f(z) = z^\alpha$ and $g(z) = z^\rho$ as in the isoelastic setting. Denote period τ consumption of good i by $x_{i\tau}$. I want to ask for the risk aversion of the decision maker with respect to the first commodity, assuming that the second commodity is fixed to some level $\bar{x}_2 = x_{21} = x_{22}$. In that case, the representation can be transformed as follows:

$$\left\{\frac{1}{2}\,(x_{11}^{\gamma_1}\bar{x}_2^{\gamma_2})^\rho + \frac{1}{2}\left[\int dp_2\,(x_{12}^{\gamma_1}\bar{x}_2^{\gamma_2})^\alpha\right]^{\frac{\rho}{\alpha}}\right\}^{\frac{1}{\rho}}$$

$$= \bar{x}_2^{\gamma_2}\left\{\frac{1}{2}\,x_{11}^{\gamma_1\rho} + \frac{1}{2}\left[\int dp_2\,x_{12}^{\gamma_1\alpha}\right]^{\frac{\rho}{\alpha}}\right\}^{\frac{1}{\rho}}$$

which is ordinally equivalent to

$$\left\{\frac{1}{2}\,x_{11}^{\gamma_1\rho} + \frac{1}{2}\left[\int dp_2\,x_{12}^{\gamma_1\alpha}\right]^{\frac{\gamma_1\rho}{\gamma_1\alpha}}\right\}^{\frac{1}{\gamma_1\rho}} . \qquad (7.2)$$ [7]

The representation in the last line characterizes preferences over the first commodity, given an arbitrary but fixed consumption level of the second good. Considering only choice over the first good, I can identify the coefficient of relative risk aversion with $\text{RRA}_1 = 1 - \gamma_1\alpha$ and the parameter of intertemporal substitutability with $\gamma_1\rho$, implying an elasticity of intertemporal substitution $\sigma_1 = \frac{1}{1-\gamma_1\rho}$. On the other hand, fixing the consumption level of the first good, yields, by the same reasoning, a coefficient of intertemporal risk aversion $\text{RRA}_2 = 1 - \gamma_2\alpha$ and an intertemporal elasticity of substitution of $\sigma_2 = \frac{1}{1-\gamma_2\rho}$. This simple example shows that the definition of risk aversion and

[7] In the case of positive discounting and a discount factor β (compare chapter 9), equation (7.2) would write as $\left\{\frac{1}{1+\beta}\,x_{11}^{\gamma_1\rho} + \frac{\beta}{1+\beta}\,\left[\int dp_2\,x_{12}^{\gamma_1\alpha}\right]^{\frac{\gamma_1\rho}{\gamma_1\alpha}}\right\}^{\frac{1}{\gamma_1\rho}}$.

intertemporal substitutability carried out above is good-specific. In the setting with two consumption commodities, there is an intertemporal substitutability for good one that, in general, differs from the intertemporal substitutability of good two, and a coefficient of relative risk aversion for good one that, in general, also differs from the coefficient of relative risk aversion for the second good. *In general, the definitions of risk aversion and intertemporal substitutability introduced in this section depend on the good under observation and its measure-scale.*[8] In such a framework (or rather with such a wording), it can even happen that the decision maker is risk averse with respect to lotteries over the first good, but risk loving with respect to lotteries over the second good. In the scenario above, the parametrization $\gamma_1 = \frac{1}{4}, \gamma_2 = \frac{3}{4}$ and $\alpha = 2$, yields such a result with $\text{RRA}_1 = 1 - \gamma_1\alpha = \frac{1}{2}$ and $\text{RRA}_2 = 1 - \gamma_2\alpha = -\frac{1}{2}$. Note that in the extension of atemporal risk aversion to multiple commodities, as developed by Kihlstrom & Mirman (1974), this finding corresponds to a decision maker, who has a positive risk premium for lotteries of one good, but a negative risk premium for lotteries over another good. Personally, I consider such a diverge of risk attitude between different commodities as unsatisfactory. My *semantic* understanding of risk aversion (or attitude towards uncertainty) asks for a measure that is not coupled to a particular consumption commodity, but rather to preference in general. In the following, such a measure is identified. To this end, I exploit the gauge freedom of the representation.

In the formalism worked out in chapter 6, both transformations carried out in the two preceding paragraphs can be interpreted as a change of the underlying Bernoulli utility function. In the first case, where I have analyzed a change of measure-scale, this has been used explicitly. In the situation of two consumption commodities, the underlying reasoning is as follows. Instead of $u(x_1, x_2) = x_1^{\gamma_1} x_2^{\gamma_2}$, I also can choose the strictly monotonic transformation $\tilde{u}(x_1, x_2) = x_1 x_2^{\gamma_2/\gamma_1}$ as the Bernoulli utility function for the representation, which is linear in the first consumption commodity. In order to depict the same preferences $\succeq$ by means of $\tilde{u}$, I have to change f and g according to lemma 1. The strictly monotonic transformation that satisfies $\tilde{u} = s \circ u$ is $s(z) = z^{1/\gamma_1} \Leftrightarrow s^{-1}(z) = z^{\gamma_1}$. Therefore, the new parametrization of risk aversion is $f \circ s^{-1}(z) = (z^{\gamma_1})^\alpha = z^{\gamma_1\alpha}$ which renders the coefficient of relative risk aversion $\text{RRA}_1 = 1 - \gamma_1\alpha$. The same reasoning applies to intertemporal aggregation rendering the intertemporal substitutability $\sigma_1 = \frac{1}{1-\gamma_1\rho}$. Similarly, choosing Bernoulli utility linear in the consumption quantity of the second good renders the coefficients of relative risk aversion $\text{RRA}_2 = 1 - \gamma_2\alpha$ and the elasticity of intertemporal substitution $\sigma_2 = \frac{1}{1-\gamma_2\rho}$. Let me conclude that commodity and

[8] Note that for more general preference settings RRA_1 will not only depend on the consumption level of the first commodity, but also on the consumption level of the fixed second good.

scale dependence of the parameters of risk aversion and intertemporal substitutability correspond to a dependence of these parameters on the choice of Bernoulli utility.

The connection between commodity and scaling dependence of the risk aversion parameter and the choice of Bernoulli utility, suggests that a notion of risk aversion which is to be independent of a specific consumption commodity, should not depend on the choice of the Bernoulli utility function. In this spirit, the following lemma identifies a natural candidate for such a measure. Denote for any $f \in \mathcal{C}^0(U)$ by $\hat{f} = \{\mathfrak{a} f : \mathfrak{a} \in \mathsf{A}\}$ the class of all members of $\mathcal{C}^0(U)$ that coincide with f up to nondegenerate affine transformations. Inverting each member of $\hat{f}$ yields the set $\hat{f}^{-1} = \{f^{-1}\mathfrak{a} : \mathfrak{a} \in \mathsf{A}\}$.[9] For $f, g \in \mathcal{C}^0(U)$, define the composition of $\hat{f}$ and $\hat{g}^{-1}$ as the class of all compositions of elements from $\hat{f}$ with elements of $\hat{g}^{-1}$, i.e. $\hat{f} \circ \hat{g}^{-1} = \{\mathfrak{a} f \circ g^{-1} \mathfrak{a}' : \mathfrak{a}, \mathfrak{a}' \in \mathsf{A}\}$. Note that $\hat{f} \circ \hat{f}^{-1} = \hat{\mathrm{id}}$. Obviously, the 'quantity' $\hat{f} \circ \hat{g}^{-1}$ denotes the class of all compositions $f \circ g^{-1}$ that go along with the same representation of $\succeq$ for a given Bernoulli utility function in the sense of theorem 2. However, there is more to it.

Lemma 2: In the representation of theorem 2, the 'quantity' $\hat{f} \circ \hat{g}^{-1}$ is gauge invariant, i.e. it is independent of the choice of the Bernoulli utility function and uniquely determined by $\succeq$.

In the light of the preceding discussion of atemporal risk aversion and intertemporal substitutability, lemma 2 states that the 'difference' between the attitude with respect to risk and with respect to intertemporal substitutability is independent of the particular good under observation and its measurement. In the isoelastic two commodity example with $f(z) = z^\alpha$ and $g(z) = z^\rho$ discussed above, this 'difference' $f \circ g^{-1}$ becomes $f \circ g^{-1}(z) = (z^\alpha)^{\frac{1}{\rho}} = z^{\frac{\alpha}{\rho}}$ where $\frac{\alpha}{\rho} = \frac{1-\mathrm{RRA}}{1-\frac{1}{\sigma}}$. The lemma implies that the same result for $f \circ g^{-1}$ should be obtained, when extracting the information on risk aversion and intertemporal substitution from equation (7.2). The latter was considering changes (only) in the first consumption commodity. It found a respective parametrization of risk aversion and intertemporal substitutability corresponding to $f(x) = x^{\gamma_1 \alpha}$ and $g(x) = x^{\gamma_1 \rho}$. As desired, this brings about $f \circ g^{-1}(z) = (z^{\gamma_1 \alpha})^{\frac{1}{\gamma_1 \rho}} = z^{\frac{\alpha}{\rho}}$. Similarly the result holds, when only looking at changes of the second consumption commodity rendering $f \circ g^{-1}(z) = (z^{\gamma_2 \alpha})^{\frac{1}{\gamma_2 \rho}} = z^{\frac{\alpha}{\rho}}$. The implications of the fact that $f \circ g^{-1}$ is uniquely determined only up to affine transformations will be discussed in section 7.4. In the following, section 7.2 works out the interpretation of the 'quantity' $f \circ g^{-1}$.

[9] Be aware that, in general, $\hat{f}^{-1} = \{f^{-1}\mathfrak{a} : \mathfrak{a} \in \mathsf{A}\}$ is not the same as $\widehat{f^{-1}} = \{\mathfrak{a} f^{-1} : \mathfrak{a} \in \mathsf{A}\}$.

7.2 Intertemporal Risk Aversion

This section characterizes the invariant quantity found in Lemma 2 axiomatically. Aiming at the representation of theorem 2, the axiomatic characterization below is for a decision maker, who has stationary preferences and a zero rate of time preference. Chapter 8 presents the general non-stationary setting. I consider a non-discounting decision maker to exhibit *weak intertemporal risk aversion*, if and only if, the following axiom is satisfied:

A6 $^{\text{w}}_{\text{nd}}$ (weak intertemporal risk aversion, no discounting) For all $\bar{x}, x_1, x_2 \in X$ holds

$$(\bar{x}, \bar{x}) \sim_F (x_1, x_2) \quad \Rightarrow \quad \bar{x} \;\succeq_T\; \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2. \tag{7.3}$$

The superscript 'w' at the axiom's label abbreviates 'weak' as opposed to 's' for 'strict', while the subscript 'nd' denotes the absence of discounting. Similarly, I define the non-discounting decision maker to exhibit *strict intertemporal risk aversion*, if and only if:

A6 $^{\text{s}}_{\text{nd}}$ (strict intertemporal risk aversion, no discounting) For all $\bar{x}, x_1, x_2 \in X$ holds

$$(\bar{x}, \bar{x}) \sim_F (x_1, x_2) \;\wedge\; x_1 \not\sim_T x_2 \quad \Rightarrow \quad \bar{x} \;\succ_T\; \frac{1}{2}x_1 + \frac{1}{2}x_2. \tag{7.4}$$

I start with the interpretation of the strict axiom. The first part of the premise in equation (7.4) states that a decision maker is indifferent between a certain constant consumption path delivering the same outcome $\bar{x}$ in both periods and another certain consumption path which delivers outcome x_1 in the first and outcome x_2 in the second period. The second part of the premise requires the consumption path (x_1, x_2) to exhibit variation in the sense that either x_1 is preferred to x_2 or vice versa.[10] Note that this relation implies[11] that either $x_1 \succ_T \bar{x}$ and $\bar{x} \succ_T x_2$ or that $\bar{x} \succ_T x_1$ and $x_2 \succ_T \bar{x}$. I want to stress that, *in the decision problem expressed on the left hand side of equation* (7.4)*, the individual knows that in either case he gets all of the chosen outcomes with certainty* (each at its time). Coming to the right hand side of equation (7.4), axiom A6$^{\text{s}}_{\text{nd}}$ demands that, for consumption satisfying the above conditions, the decision maker should prefer the consumption of $\bar{x}$ with certainty over a lottery yielding either x_1 or x_2, each with probability a half. The intuition is that he values x_1 and x_2 differently.

[10]Note that the axiom compares the first period outcome x_1 directly with x_2, by taking it as a second period outcome. This switching of the periods does not pose any problems, as the decision maker has stationary preferences and a discount rate of zero. See also footnote 12. The axiomatic formulation of intertemporal risk aversion for the general case avoids this switch of period. See chapter 8, pages 117 and 120, for the general formulation and chapter 127, page 139, for the particular case of stationary preferences and positive discounting.

[11]In combination with the other axioms given in the representation theorem 2.

Therefore, *the lottery might make him better off, but with the same probability it can make him worse off. In the latter case, and in contrast to the decision problem in the premise, he will never receive the higher outcome.* Let me point out that the comparison of the intertemporal consumption paths in the premise is used to calibrate the 'worse off' and the 'better off' outcomes in the lottery with respect to each other.[12]

Calling preferences satisfying axiom $A6^{s}_{nd}$ *intertemporally risk averse* is motivated by the following (interrelated) reasons. First, without acknowledging a trade-off over time, the concept could not be defined. Second, the naming responds to the fact discussed in the previous section that the conventional definition of attitude towards risk describes what Normandin & St-Amour (1998, 268) call atemporal risk aversion. In comparison the concept of intertemporal substitutability describes the attitude of the decision maker to trade consumption over time. In this sense the theorem below delivers the third motivation for the naming. It will reveal that intertemporal risk aversion is characterized simultaneously by the functions f and g and, thus, combines information on atemporal risk attitude with the intertemporal characteristics of preference.

The interpretation of the *weaker axiom* $A6^{w}_{nd}$ is analogous to that of axiom $A6^{s}_{nd}$, except that the consumption path (x_1, x_2) is allowed to coincide with $(\bar{x}, \bar{x})$, and the implication only requires that the lottery is not strictly preferred to the certain consumption path. If axiom $A6^{s}_{nd}$ ($A6^{w}_{nd}$) is satisfied with $\succ_F$ ($\succeq_F$) replaced by $\prec_F$ ($\preceq_F$), I call the decision maker a strong (weak) *intertemporal risk seeker.* If his preferences satisfy weak intertemporal risk aversion as well as weak intertemporal risk seeking, the decision maker is called *intertemporally risk neutral.* The following theorem relates the concept of intertemporal risk aversion to the invariant found in lemma 2 of the preceding section.[13]

Theorem 3: Let the triple (u, f, g) represent the set of preferences $\succeq$ in the sense of theorem 2. Then the following assertions hold:

a) A decision maker is strictly intertemporally risk averse [seeking] in the sense of axiom $A6^{s}_{nd}$, if and only if, $f \circ g^{-1}(z)$ is strictly concave [convex] in $z \in \Gamma$ for an increasing version of f.[14]

b) A decision maker is weakly intertemporally risk averse [seeking] in the sense of

[12] Note that the interpretation of axiom $A6^{s}_{nd}$ depends crucially on the fact that the decision maker is not discounting future consumption. Otherwise the constellation in equation (7.3) could be entirely due to x_2 being a very bad outcome that had been highly discounted in the premise. While axiom $A6^{s}_{nd}$ at this point is a simplified version to focus on the essential idea, it will gain its own standing in chapter 10.4, where I give an axiomatization that implies a zero discount rate.

[13] Recall the definition $\Gamma = (\underline{G}, \overline{G})$.

[14] Which is equivalent to strict convexity [concavity] for a decreasing choice of f.

axiom $A6^{w}_{nd}$, if and only if, $f \circ g^{-1}(z)$ is concave [convex] in $z \in \Gamma$ for an increasing version of f.
c) A decision maker is intertemporally risk neutral, if and only if, $\hat{f} = \hat{g}$. In that case the decision maker maximizes intertemporally additive expected utility.

To interpret the result, note that $f \circ g^{-1}$ concave can be paraphrased as f being concave with respect to g (Hardy et al. 1964). Assume for example that $f \circ g^{-1} = s$ and s is some concave function. Then the relation is equivalent to $f = s \circ g$ so that f is a concave transformation of g, or expressed intuitively f is 'more concave' than g. Reconsidering the discussion of section 7.1, the interpretation in terms of the Epstein Zin gauge would be as follows. Concavity of f characterizes the decision maker's degree of (atemporal) risk aversion, while concavity of g characterizes his desire to smooth out consumption over time. Hence, in light of the above, $f \circ g^{-1}$ concave receives the interpretation of the decision maker being more averse to substitute consumption into a risky state than to substitute it into a certain future. To express this statement slightly different, think of a situation, where a decision maker has the chance to either smooth out consumption over time or over risk. Then, whenever the intertemporally risk neutral decision maker is indifferent between the two options, the intertemporally risk averse decision maker prefers to smooth out consumption over the risky states, while the intertemporally risk seeking decision maker prefers to keep the risk but smooth out consumption over time. In section 7.1 it has been shown that the 'quantity' $\hat{f} \circ \hat{g}^{-1}$ does not depend on the gauge and, thus, on a particular commodity and its measure scale. Hence, the reasoning in terms of the difference between uncertainty aggregation characteristics and attitude towards intertemporal substitution is more general than the individual interpretation of the terms f and g in the Epstein Zin gauge, which is tied to a particular commodity and its measure scale. I think that the best economic interpretation of $f \circ g^{-1}$ is the one laid out in axioms $A6^{w}_{nd}$ and $A6^{s}_{nd}$. The subsequent section interprets the axioms and the expression $f \circ g^{-1}$ in terms of risk aversion with respect to welfare gains and losses, and relates it to the idea of precaution as developed in sections 5.2 and 5.3. Before doing so, I end this section with a remark on the determinacy of the concavity of $f \circ g^{-1}$.

The observant reader might have noticed that the characterization of intertemporal risk aversion in theorem 3 relies on $f \circ g^{-1}$, while only the 'quantity' $\hat{f} \circ \hat{g}^{-1}$ is independent of the choice of Bernoulli utility. Therefore, concavity of $f \circ g^{-1}$ should not depend on affine transformations of the function or its argument. This fact is verified in the following proposition.

Proposition 9: If some function $h \in \hat{f} \circ \hat{g}^{-1}$ is (strictly) concave, then all functions in $\hat{f} \circ \hat{g}^{-1}$ are (strictly) concave. If some function $h \in \hat{f} \circ \hat{g}^{-1}$ is (strictly) convex, then all functions in $\hat{f} \circ \hat{g}^{-1}$ are (strictly) convex.

Proposition 9 shows that the fact of being intertemporally risk averse or risk seeking does not depend on the affine freedom of $f \circ g^{-1}$. To succeed in defining a quantitative measure of intertemporal risk aversion, however, the affine freedom in $f \circ g^{-1}$ requires a little more thought.[15] Section 7.4 will be dedicated to that question.

7.3 Welfare and Precaution

Technically, relating intertemporal risk aversion and precaution is merely a task of moving into the certainty additive gauge ($g = \text{id}$). Then the representation given in corollary 3 corresponds to the setup of sections 5.2 and 5.3.[16] Moreover, in the certainty additive gauge it is $f \circ g^{-1} = f$. Therefore, assertion a) of theorem 3 states that, for an increasing choice of f, strict intertemporal risk aversion is equivalent to a strictly concave function f. But proposition 6 on page 69 identified the same condition as a characterization of precaution. This simple reasoning is the technical content of this section. However, a closer look at the $g = \text{id}$ representation yields an interesting interpretation of the concept of intertemporal risk aversion developed in the preceding section.

Let me start out by recalling that the cardinality of the welfare function in sections 5.2 and 5.3 relied entirely on some 'exogenous intuition' of welfare being something cardinal. Technically, it dropped from heaven. With the background developed in chapter 6, the cardinality of u is implicit to the formal setup. Evaluation of the threat of harm scenario with equation (5.3) corresponds to the $g = \text{id}$-gauge. In this gauge, choice over certain consumption paths is immediately characterized by the sum over per period utility u. If the sum of per period utility is the same, then the overall welfare is the same, in the sense that indifference between the corresponding consumption paths prevails. In other words, an amount Δu less in one period can be compensated by the same amount Δu more in another period. The same reasoning and, thus, interpretation of Bernoulli utility generally fails for other gauges. Therefore, I think that the Bernoulli utility in the certainty additive gauge most closely relates to the semantics of the word welfare. At the end of this section, I will come back to this idea and attempt to further motivate, why I think that the $g = \text{id}$-gauge, and the interpretation of the corresponding Bernoulli utility as welfare, is a particularly helpful representation of a decision problem. For now,

[15]It is the affine transformation $\tilde{\mathfrak{a}}$ in $fg^{-1} \rightarrow \mathfrak{a} fg^{-1} \tilde{\mathfrak{a}}$ that requires some special attention.

[16]Compare in particular equation (5.3) on page 69.

let me use the interpretation of *welfare as certainty additive Bernoulli utility* to revisit the concept of intertemporal risk aversion. For this purpose, I denote the particular Bernoulli utility function that corresponds to the $g =$ id-gauge as welfare u^{welf}. It is the notion of welfare that has been applied implicitly already in the discussion of sections 5.2 and 5.3.

Now, I give a reinterpretation of axioms $A6^{\text{w}}_{\text{nd}}$ and $A6^{\text{s}}_{\text{nd}}$. Applying the notion of welfare introduced in the preceding paragraph, the first part of the premise in axiom $A6^{\text{s}}_{\text{nd}}$ corresponds to the requirement that welfare of the two consumption paths adds up to the same overall welfare, i.e. $u^{\text{welf}}(x_1) + u^{\text{welf}}(x_2) = u^{\text{welf}}(\bar{x}) + u^{\text{welf}}(\bar{x})$. The second part of the premise in axiom $A6^{\text{s}}_{\text{nd}}$ adds the demand that the welfare evaluation of outcome x_1 and x_2 should not coincide. Without loss of generality (wlog), assume that $u^{\text{welf}}(x_1) > u^{\text{welf}}(x_2)$. Then, the *potential welfare gain in the lottery* giving with equal probability either x_1 or x_2 instead of the certain alternative $\bar{x}$, is *just as big as* the *potential welfare loss*, i.e. $u^{\text{welf}}(x_1) - u^{\text{welf}}(\bar{x}) = u^{\text{welf}}(\bar{x}) - u^{\text{welf}}(x_2) = \Delta u^{\text{welf}}$. Hence, in this notion, intertemporal risk aversion demands that a certain welfare level should be preferred to a welfare lottery that renders a welfare gain of Δu^{welf}, with probability a half, and a welfare loss of Δu^{welf}, also with probability a half. With this interpretation, *intertemporal risk aversion can be understood as* risk aversion with respect to welfare gains and losses or just as *risk aversion on welfare.*

In this notion, *precautionary behavior in the sense of sections 5.2 and 5.3 corresponds to strict risk aversion with respect to welfare gains and losses.* A straight forward intuition that was appealed to already in the introductory discussion, and that is excluded in the preference framework represented by intertemporally additive expected utility. In the setup of chapters 5.2 and 5.3 a decision maker, who exhibits risk aversion with respect to welfare gains and losses as described above, will have a higher willingness to reduce first period welfare in order to avoid a future threat of harm than does a decision maker evaluating by intertemporal expected utility, as characterized in equation (5.1). Note that if the precautionary decision maker has the chance to reduce the risk in the second period by giving up some of his welfare in the first, his ex-post welfare evaluated on the ex-post certain consumption paths will, on average, be lower than that of an intertemporally risk neutral decision maker with the same welfare evaluation over certain consumption paths. However, it is the very nature of risk aversion that evaluation of the future is not simply based on the average outcome, but also on the 'certainty' with which the *average* outcome (or an outcome close to the average) actually takes place in the *individual occasion.* This characteristic of risk aversion corresponds to a cost (on average) that comes along with precautionary behavior. Therefore, the *trade-off* between precautionary action and intertemporally risk neutral action is that of avoiding risk at the price of being worse off on average.[17] Recall from the discussion in chapter 5.2 that the arbitrariness with respect to its application, was a major criticism directed against the precautionary *principle*. The formulation of precaution as strict intertemporal risk aversion eliminates this arbitrariness by translating it into the question of specifying a degree of intertemporal risk aversion. *Once such a degree of risk aversion is specified, the rule when to undergo precautionary action does not depend on the particular class of outcomes (or field of policy) the decision maker is facing. This procedure offers a principled approach to balance costs against benefits and, at the same time, allows for precautionary behavior that is excluded in a cost-benefit analysis based on a preference framework corresponding to intertemporally additive expected utility.* In the next section, I will set up such a quantitative measure for the degree of intertemporal risk aversion. Again, the representation in the $g = \text{id}$-gauge and the corresponding interpretation as risk aversion on welfare will prove useful for understanding the indeterminacy of the expression $f \circ g^{-1} \in \hat{f} \circ \hat{g}^{-1}$ and its implications for such a measurement. Before presenting the corresponding analysis, let me close the section with suggesting a reasoning, *why* the $g = \text{id}$-gauge might be a particularly convenient representation to think about intertemporal decision problems.

From a mathematical point of view both, the certainty additive gauge (where $g = \text{id}$), and the Kreps Porteus gauge (where $f = \text{id}$), have a special appeal as either of them makes one of the two aggregators $\mathcal{N}^g$ or M^f additive.[18] Contemplating why either of the two gauges might be preferable and in what sense, let me go back to the work of von Neumann & Morgenstern (1944). Before the authors introduce their famous representation theorem, they discuss the usefulness of yielding an additive representation. They explain that calling two natural operations addition "is not intended as a claim that the two operations with the same name are identical, [...] it only expresses the opinion that they possess similar traits, and the hope that some correspondence between them will ultimately be established." (von Neumann & Morgenstern 1944, 21). As analogies

[17] Note that, here, the notion of an average appeals to the frequency definition of probability. As discussed in chapter 5.2, probabilistic reasoning applies to a much wider area of uncertainty. In particular with respect to climate change and corresponding events like a shut down of the northern arm of the Atlantic Conveyor Belt or a disintegration of the West Antarctic ice sheet, a frequency definition seems a little far-fetched. However, in these situations it appears to me at least as reasonable not to base the individual decision on the purely mathematical notion of an average outcome.

[18] In the general multi-commodity setting, u cannot be chosen linear in (all) consumption, as in the Epstein Zin gauge for one-commodity. At most, one good and its perfect substitutes could be picked under the assumption that nonsatiation in the interior of X holds for this commodity class. See section 7.1.

for additive quantities and similarity of traits, they discuss the concepts of mass and energy.[19]

In the preference framework described here, I have to conduct two aggregation operations, one over time and another over uncertainty. It was shown that, in general, only one of them can be rendered additive. Now let me look at the analogies mentioned by von Neumann & Morgenstern (1944, section 3.4), namely mass and energy and their similarity in trait. When thinking about mass as an additive operation, I have in mind the property that gaining one pound next week, another pound in the second and loosing two pounds in the third will bring me back to my original weight. Energy might even prove a better example for making this point, as it is an abstract quantity, introduced by physicists, to describe some regularity that cannot be observed directly - just as utility. Furthermore, energy is defined in order to bring this regularity into a form that makes it possible to think about it in a similar way as about the physically observable mass, that is, in terms of adding and subtracting energy at different points of time.[20] Motivated by the discussion above, I suggest to introduce the quantity named *welfare* as the special Bernoulli utility that shares the similarity with the above quantities of making welfare comparisons additive over time. Then, in the certainty additive gauge, a unit of welfare Δu^{welf} more in period one can compensate a unit Δu^{welf} of welfare less in the second period. In a general gauge, this characterisitc of additivity over time on certain consumption paths is not satisfied by Bernoulli utility itself, but by the composed function $g \circ u$ which equals u^{welf} up to an affine transformation.[21] More precisely, u^{welf} itself is only unique up to affine transformations (see corollary 3). A possibility to render welfare measurement in the sense of a certainty additive Bernoulli utility function u^{welf} unique, is by fixing exogeneously two welfare levels, e.g. the welfare of the worst and the best outcome(s). This step is equivalent to fixing the range of u^{welf} or $g \circ u$ to a given nondegenerate closed interval W^*.

[19]Let me add that additivity is a concept that emerged from the calculus of natural numbers. Mathematically, it is therefore based on the Peano axioms. From a more 'physical' perspective, it is based on the observation of similar rules for different classes of materialistic objects, describing what happens when putting (adding) and taking away objects of the same type. This structure is inherent also in von Neumann & Morgenstern's (1944) examples of mass and energy.

[20]While stressing the similarity in trait between additivity of mass, energy and welfare be aware that the analogy, like any analogy, also has its limits.

[21]From the first part of the proof of representation theorem 2 it is obvious that by construction it is $\widehat{g \circ u} = \widehat{\text{id} \circ u^{\text{welf}}} = \widehat{u^{\text{welf}}}$, where u^{welf} corresponds to the particular Bernoulli utility function which appears in equation (6.3) of axiom A4.

7.4 Measures of Intertemporal Risk Aversion

In this section, I establish a measure that quantifies a degree of intertemporal risk aversion. The natural candidate is the construction of an analogue to the coefficient of relative risk aversion in the atemporal setting. For a twice differentiable function $f \circ g^{-1} : \Gamma \to \mathrm{I\!R}$ with $\Gamma \subset \mathrm{I\!R}$, I define such a measure of *relative intertemporal risk aversion* as the function:

$$\mathrm{RIRA} : \Gamma \to \mathrm{I\!R}$$
$$\mathrm{RIRA}(z) = -\frac{(f \circ g^{-1})''(z)}{(f \circ g^{-1})'(z)}\, z\,. \tag{7.5}$$

In the light of section 7.1, and lemma 2 I have to check whether the definition of RIRA is invariant under affine transformations of f and g. Unpleasantly, this is not the case. In other words, the coefficient[22] of relative intertemporal risk aversion is not uniquely determined by the underlying preference relation. While the above definition of RIRA eliminates the indeterminacy corresponding to the affine freedom of f, it still depends on the particular affine specification of g.[23] In detail, define for some $\mathfrak{a}, \tilde{\mathfrak{a}} \in \mathcal{A}$ the functions $\tilde{f} = \mathfrak{a} f$ and $\tilde{g} = \tilde{\mathfrak{a}} g$ and let $\tilde{\mathfrak{a}}(z) = \tilde{a} z + \tilde{b}$. Then, for the new choice $\tilde{f} \circ \tilde{g}^{-1} = \mathfrak{a} f \circ g^{-1} \tilde{\mathfrak{a}}^{-1} \in \hat{f} \circ \hat{g}^{-1}$, defined on $\tilde{z} \in \tilde{\Gamma} = \tilde{\mathfrak{a}}\Gamma$, the coefficient of relative intertemporal risk aversion calculates as

$$\mathrm{RI\tilde{R}A}(\tilde{z}) = -\frac{\big(\tilde{f} \circ \tilde{g}^{-1}\big)''(\tilde{z})}{\big(\tilde{f} \circ \tilde{g}^{-1}\big)'(\tilde{z})}\,\tilde{z} = -\frac{\big(\mathfrak{a} f \circ g^{-1}\tilde{\mathfrak{a}}^{-1}\big)''(\tilde{z})}{\big(\mathfrak{a} f \circ g^{-1}\tilde{\mathfrak{a}}^{-1}\big)'(\tilde{z})}\,\tilde{z} = -\frac{(f \circ g^{-1})''(\tilde{\mathfrak{a}}^{-1}\tilde{z})\frac{1}{\tilde{a}^2}}{(f \circ g^{-1})'(\tilde{\mathfrak{a}}^{-1}\tilde{z})\frac{1}{\tilde{a}}}\,\tilde{z}$$
$$= -\frac{(f \circ g^{-1})''(\tilde{\mathfrak{a}}^{-1}\tilde{z})}{(f \circ g^{-1})'(\tilde{\mathfrak{a}}^{-1}\tilde{z})}\,\frac{1}{\tilde{a}}\,\tilde{z}\,.$$

Evaluating the new coefficient of relative intertemporal risk aversion $\mathrm{RI\tilde{R}A}$ at the same consumption level as the old coefficient of relative intertemporal risk aversion RIRA,

[22] I adopt the word coefficient also for the case where the function is non-constant and, thus, 'the' coefficient is a function of z.

[23] More precisely, it eliminates the affine freedom corresponding the transformation $\mathfrak{a} \in \mathcal{A}$ in the expression $\mathfrak{a} f \circ g^{-1}\tilde{\mathfrak{a}}^{-1}$ while it depends on the transformation $\tilde{\mathfrak{a}} \in \mathcal{A}$. Only in the representation of theorem 2, the freedom corresponding to the transformation $\mathfrak{a}$ corresponds to the indeterminacy of f and the the freedom corresponding to the transformation $\tilde{\mathfrak{a}}$ corresponds to the indeterminacy of g. For the g-gauge both indeterminacies are due to the freedom of f, as can be observed from the moreover part of corollary 3. Similarly for the f-gauge both indeterminacies are due to the freedom of g, as can be observed from the moreover part of corollary 2.

which corresponds to $\tilde{z} = \tilde{\mathfrak{a}}z$,[24] the following result is found

$$\mathrm{RI\tilde{R}A}(\tilde{z})\Big|_{\tilde{z}=\tilde{a}z+\tilde{b}} = -\frac{\big(\tilde{f}\circ\tilde{g}^{-1}\big)''(\tilde{z})}{\big(\tilde{f}\circ\tilde{g}^{-1}\big)'(\tilde{z})}\,\tilde{z}\Bigg|_{\tilde{z}=\tilde{a}z+\tilde{b}} = -\frac{(f\circ g^{-1})''(z)}{(f\circ g^{-1})'(z)}\,\frac{\tilde{a}\,z+\tilde{b}}{\tilde{a}}\,. \tag{7.6}$$

As claimed above, the affine indeterminacy corresponding to the transformation $f \circ g^{-1} \to \mathfrak{a} f \circ g^{-1}$ leaves the coefficient of relative intertemporal risk aversion unchanged. However, an affine change corresponding to $\tilde{b}$ in $f \circ g^{-1} \to f \circ g^{-1}\tilde{\mathfrak{a}}^{-1}$ changes the coefficient.

An analogous result holds, when defining the *coefficient of absolute intertemporal risk aversion* as the function:

$$\mathrm{AIRA} : \Gamma \to \mathrm{I\!R}$$

$$\mathrm{AIRA}(z) = -\frac{(f\circ g^{-1})''(z)}{(f\circ g^{-1})'(z)}\,. \tag{7.7}$$

Then, changing $f \circ g^{-1} \to \tilde{f} \circ \tilde{g}^{-1}$ like above and evaluating the new coefficient of absolute intertemporal risk aversion for the same consumption level, yields

$$\mathrm{AI\tilde{R}A}(\tilde{z})\Big|_{\tilde{z}=\tilde{a}z+\tilde{b}} = -\frac{\big(\tilde{f}\circ\tilde{g}^{-1}\big)''(\tilde{z})}{\big(\tilde{f}\circ\tilde{g}^{-1}\big)'(\tilde{z})}\Bigg|_{\tilde{z}=\tilde{a}z+\tilde{b}} = -\frac{(f\circ g^{-1})''(z)}{(f\circ g^{-1})'(z)}\,\frac{1}{\tilde{a}}\,. \tag{7.8}$$

Again, the affine indeterminacy corresponding to the transformation $f \circ g^{-1} \to \mathfrak{a} f \circ g^{-1}$ leaves the coefficient of absolute intertemporal risk aversion unchanged. However, a linear change corresponding to $\tilde{a}$ in $f \circ g^{-1} \to f \circ g^{-1}\tilde{\mathfrak{a}}^{-1}$ changes the coefficient.

I want to present an intuition for the indeterminacy of the above coefficients of intertemporal risk aversion. For this purpose, note that a specification of $f \circ g^{-1}$ together with $g \circ u$, completely determines the underlying preference relation.[25] However, this is no longer true for a specification of $g \circ u$ and $f \circ \hat{g}^{-1}$ (or $\hat{f} \circ \hat{g}^{-1}$). The reason is as follows. The function g itself is only determined up to affine transformations and the particular specification entering the representation does not matter. Yet, it has to be the same function g that enters the expression of intertemporal risk aversion $f \circ g^{-1}$ and the expression that determines choice on certain consumption paths $g \circ u$. As long as both are changed simultaneously corresponding to the same affine transformation $\mathfrak{a} \in \mathsf{A}$, i.e. $f \circ g^{-1} \to f \circ g^{-1}\tilde{\mathfrak{a}}^{-1}$ and $g \circ u \to \tilde{\mathfrak{a}} g \circ u$, the represented underlying set of preference relations $\succeq$ stays the same. But setting up the evaluation functional, when

[24] The argument z of the function $f \circ g^{-1}$ in the representation is a weighted arithmetic mean of two entries scaling with g.

[25] To see that $f \circ g^{-1}$ together with $g \circ u$ completely specifies $\succeq$, just note that $f \circ u = (f \circ g^{-1}) \circ (g \circ u)$ and, hence, all terms in the following strictly monotonic transformation of the representation $\mathcal{N}^g\Big[u(x)\,,\,\mathcal{M}^f(p,u)\Big]$ are specified: $f \circ g^{-1}\left[\frac{1}{2}\, g \circ u(x) + \frac{1}{2}\, g \circ f^{-1}\left[\int dp^{x_2}\, f \circ u(x_2)\right]\right]$.

given only the information $f \circ \hat{g}^{-1}$ and $g \circ u$, it is unknown which element of the class $f \circ \hat{g}^{-1}$ corresponds to the same g as in $g \circ u$. Combining $g \circ u$ with an arbitrary element $f \circ \tilde{g} \in f \circ \hat{g}$, however, yields in general an evaluation different from that implied by $g \circ u$ in combination with $f \circ g^{-1}$. In other words, *the affine freedom of g does not translate into a freedom of the measure for intertemporal risk aversion that is independent of the specification of the evaluation over certain consumption paths.* A particular specification of the expression $g \circ u$, evaluating certain outcomes, has to go a along with a particular choice of the risk measure.

An example might help to understand the preceding reasoning and relate it to the discussion on measure-scale in section 7.1. Imagine an employee in a small enterprise who experiences an unusual situation. His boss steps into his office and offers to play the following lottery. He flips a coin and, if head comes up, the employee gets three weeks of summer vacation instead of the usual two weeks. However, if tail comes up, the boss wants him to cut down his summer vacations to one week. The employee declines the offer. Observing the situation, I would like to find out the underlying motive of the employee to reject the offer. I have two possible theories about him, which I would like to tell apart. On the one hand, the employee might love to have a vacation of three weeks. He even thinks that the third week of vacation would be more valuable to him than the second, for example because three weeks of vacation would allow him to undertake the big travel he had been waiting for. But, at the same time he is risk averse, or, in less sophisticated words, he fears to loose in the lottery, and therefore declines it.[26] On the other hand, the employee might be risk neutral or even keen on gambling, but he values a third week of vacation so little as compared to the second that he refuses the lottery of his boss. In an atemporal world, I had a hard time to distinguish these two possibilities. However, in the intertemporal setting, a differentiation of the two motives is possible. At least, if the boss allows me to offer the following deal to the perfectly patient (i.e. non-discounting) employee.[27] I walk into the employee's office with the offer to trade in one week of this year's vacation for an extra week of vacation in the next year. If he accepts, I know that his declining of the lottery was due to risk aversion. Under certainty, he has a positive willingness to trade his second week of vacation for a third (a reasoning obviously impossible without time). As he nevertheless declined the

[26] I encourage the reader to first go over the example relying entirely on a semantic intuition of 'risk aversion' (as it prevailed before the study of von Neumann & Morgenstern's (1944) theory). Formalizing the intuitive argument corresponds to the concept of intertemporal risk aversion as introduced in section 7.2.

[27] To see how the question has to be asked for a discounting employee or someone with non-stationary preferences compare page 118 in chapter 8.

lottery offered by his boss, he must be risk averse. If the employee declines my offer, I know that he values a second week of vacation higher than the third. Of course, without further inquiries, I cannot tell whether he might in addition be risk averse. However, given that one of the two theories about the employee laid out above is true, it would be the second.

Now, think about measuring the employee's degree of risk aversion. The easiest perspective is the one discussed in section 7.3, corresponding to the certainty additive gauge, where intertemporal risk aversion can be understood as risk aversion on welfare. Here the intertemporal trade-off, i.e. the employee's willingness to trade days of vacation between one period and the next, determines the Bernoulli utility function, which I associated with the agents welfare in section 7.3. However, observing the employee's preferences over certain consumption paths only renders a welfare function that is determined up to affine transformations. This is, it does not render a natural level of 'zero welfare', nor does it deliver a natural unit to measure welfare. Now intertemporal risk aversion describes his attitude towards lotteries over welfare. But for determining a coefficient of *relative* risk aversion over welfare lotteries, I have to know the '*zero welfare*' level. Similarly, to identify the coefficient of *absolute* risk aversion on welfare, I have to know the *unit of welfare*. Not knowing the 'zero welfare' level corresponds, in equation (7.6), to the undetermined constant $\tilde{b}$ in the coefficient of relative intertemporal risk aversion. Not knowing the 'unit of welfare' corresponds, in equation (7.8), to the undetermined constant $\tilde{a}$ in the coefficient of absolute intertemporal risk aversion. Fixing a unique numerical coefficient of intertemporal risk aversion is, therefore, a question of fixing a measure-scale, just as for atemporal risk aversion in section 7.1. However, this time the measure-scale is not in terms of physical characteristics of a consumption good, but in terms of the implicitly derived concept of welfare. Analogously to most arbitrarily divisible goods, a natural unit is not given and must be fixed by *convention* (see also footnote 6). More difficult is probably the interpretation of fixing a zero welfare level, in order to render a meaning to the coefficient of *relative* risk aversion. Again, it must be a convention. Note that chapter 10 develops a reasoning based on the decision maker's attitude with respect to the timing of uncertainty resolution that implies constant absolute intertemporal risk aversion and, thus, a representation that does not depend on $\tilde{b}$ or the 'zero welfare' level.

In general, the corollaries below solve the problem of quantifying intertemporal risk aversion formally, by incorporating the information that has to come from convention. As the discussion above was mainly led within the certainty additive gauge ($g = \text{id}$), let me start out with the case where g is imposed. At first sight, it might be surprising that the affine freedom corresponding to $\tilde{\mathfrak{a}}$ in $\hat{f} \circ \hat{g}^{-1}$ still prevails in a gauge where g is exogenously fixed. However, chapter 6.4 worked out that fixing g instead of u does not abolish any of the indeterminacy. The freedom that corresponds to g in the representation of theorem 2, translates into an additional freedom of f in the $g =$ id-gauge, as worked out in the moreover part of corollary 3. For the latter representation, the pair (u, f) was only determined up to simultaneous transformations of type $(u, f) \rightarrow (\tilde{u}, \tilde{f}) = (g^{-1}{\mathfrak{a}^+}^{-1} g\ u, \mathfrak{a}\ f\ g^{-1}\mathfrak{a}^+ g)$ with $\mathfrak{a} \in \mathsf{A}$ and $\mathfrak{a}^+ \in \mathsf{A}^+$. Hence, the expression for intertemporal risk aversion in the standard g-gauge still is only determined up to transformations of type $f \circ g^{-1} \rightarrow \tilde{f} \circ g^{-1} = \mathfrak{a}\ (f\ g^{-1}\mathfrak{a}^+ g) \circ g^{-1} = \mathfrak{a}\ f \circ g^{-1}\mathfrak{a}^+$. Let me point out that, in the $g =$ id-gauge, a closer look at the allowed $\mathfrak{a}^+$ transformations of the pair (u, f) is another way to see the relation between measuring intertemporal risk aversion and fixing some freedom in the welfare measure. In this gauge, f represents intertemporal risk aversion and u was given an interpretation of welfare. Then, the transformation $(u, f) \rightarrow (\tilde{u}, \tilde{f}) = ({\mathfrak{a}^+}^{-1} u, f\ \mathfrak{a}^+)$ corresponds to a shift of $\mathfrak{a}^+$ between welfare and intertemporal risk aversion. Only when fixing two points of u by some convention, f is uniquely determined.

Corollary 4 (g^+-gauge) :

Let there be given a set of binary relations $\succeq = (\succeq_F, \succeq_T)$ and a strictly monotonic and continuous function $\boldsymbol{g}$: $\mathrm{I\!R} \rightarrow \mathrm{I\!R}$. Choose a nondegenerate closed interval $U^* \subset \mathrm{I\!R}$. The following equivalence holds:
The set of binary relations $\succeq$ satisfies

$i - iii$) of theorem 2,

if and only if, there exists a continuous and surjective function $u : X \rightarrow U^*$ as well as a strictly monotonic and continuous function $f : U^* \rightarrow \mathrm{I\!R}$ such that

$v - vi$) of theorem 2 hold.

Moreover, u is determined uniquely and f is determined up to nondegenerate affine transformations. The measures of intertemporal risk aversion RIRA and AIRA as defined in equations (7.5) and (7.7) are determined uniquely.

I call the representation of corollary 4 a g^+-gauge, because it is a g-gauge with the additional fixing of Bernoulli utility at two points. Corollary 4 is a trivial consequence of corollary 3, where the fixing of u for any two outcomes abolishes its indeterminacy. Here, I have fixed the Bernoulli utility for the worst and the best outcome by requiring that u is onto a given codomain U^*. In the $g =$ id-gauge, the latter fixes immediately the scale of evaluation for the certain outcome paths, and U^* can be interpreted as the welfare range in the sense of W^* in section 7.3. If g is not chosen as the identity, then $g \circ u$ gives the evaluation of certain consumption paths and $W^* = g(U^*)$, also uniquely fixed, would correspond to the welfare range in the sense laid out in section 7.3. Instead of fixing evaluation of the best and the worst outcome, I could as well fix a 'zero welfare' level[28] and a 'welfare unit', or Bernoulli utility for any two points that differ in their evaluation.

The same approach copes with the indeterminacy of intertemporal risk aversion in the representation of theorem 2. I call the representation a u^+-gauge as a particular choice of $u \in B_{\succeq}$ is imposed and, in addition, the range of g is fixed. The latter fixing determines again the scale, on which the evaluation over certain consumption paths (corresponding to $g \circ u$) is based.

Corollary 5 **(u^+-gauge)** :

Let there be given a set of binary relations $\succeq = (\succeq_F, \succeq_T)$ and a Bernoulli utility function $u \in B_{\succeq}$ with range U. Choose a nondegenerate closed interval $W^* \subset \mathbb{R}$. The following equivalence holds:
The set of binary relations $\succeq$ satisfies

$i - iii)$ of theorem 2,

if and only if, there exists a strictly monotonic and continuous function $f : U \to \mathbb{R}$ as well as a strictly monotonic, continuous and surjective function $g : U \to W^*$ such that

$v - vi)$ of theorem 2 hold.

Moreover, g is determined uniquely and f is unique up to nondegenerate affine transformations. The measures of intertemporal risk aversion RIRA and AIRA as defined in equations (7.5) and (7.7) are determined uniquely.

Instead of fixing the welfare range corresponding to $G = W^*$, I could again fix as well a 'zero welfare' level and a 'welfare unit', or the function g on any two points that differ in their evaluation. In both corollaries above, the risk measures depend on the choice of the imposed range of g. While in corollary 5 it is fixed directly, in corollary 4 it is determined by imposing $g : \mathbb{R} \to \mathbb{R}$ together with the domain U so that $G = g(U) = W^*$ is fixed. The following lemma states more generally that all what is needed for the uniqueness of the risk measures RIRA and AIRA, is the set of preference relations and the range of the function g in any of the representations.

[28] At least, as long as it should correspond to some outcome within the choice set X. See chapters 8.2 and 9.4 for a formalization of fixing the 'zero welfare level' level or the 'welfare unit'.

Lemma 3: Let there be given a set of preference relations $\succeq = (\succeq_F, \succeq_T)$ satisfying axioms A1-A5 as well as a closed nondegenerate interval $W^* \subset \mathrm{I\!R}$. For any representation in the sense of theorem 2 where g is onto $G = W^*$, the risk measures RIRA *and* AIRA *are determined uniquely and independent of the choice of Bernoulli utility.*

Lemma 3 states that, once the scale for welfare in the sense of section 7.3 is fixed by imposing its range, the measures of intertemporal risk aversion RIRA and AIRA are uniquely specified by the preference relations.[29] Then, the choice of Bernoulli utility in the representation, corresponding to different good-specific measures of atemporal risk aversion, does not affect the measures of intertemporal risk aversion.

7.5 Summary

I have introduced a notion of general and precautionary uncertainty aggregation rules. In this notion, the expected value operator corresponds to the particular uncertainty aggregation rule that is linear in the evaluation of outcomes. I have elaborated that in an *atemporal* setting, the use of different uncertainty aggregation rules is equivalent to a change in the Bernoulli utility function evaluating (certain) outcomes. In an *intertemporal* framework, however, aggregation of value over time adds an additional dimension to the evaluation problem. I have shown that choosing Bernoulli utility in a way yielding a linear uncertainty aggregation implies, in general, a nonlinear aggregation of value over time. Considering additivity of welfare over time as a useful attribute for an economic interpretation of a preference representation, I have offered an alternative representation which keeps linearity in welfare over time, and uses more general uncertainty aggregation rules for the evaluation of uncertainty. In such a setting, the class of precautionary uncertainty aggregation rules has been shown to capture an important concern of the *precautionary principle*. These rules imply a higher willingness to undergo preventive measures in order to avoid a threat of harm, than does an evaluation based on the standard model of intertemporally additive expected utility.

I have worked out the general time consistent model supported by the von Neumann-Morgenstern axioms for choice under uncertainty, and additive separability over time for certain consumption paths. In this framework, I have introduced a new notion of risk aversion. Due to its crucial dependence on the intertemporal structure of pref-

[29] Only in the certainty additive gauge the intertemporal trade-off measure that I interpreted as welfare coincides with Bernoulli utility, in general it corresponds to range$(g) = G$.

erences, I labeled it *intertemporal risk aversion.* I have provided an axiomatic foundation and worked out measures of absolute and relative intertemporal risk aversion. These measures were seen to depend only on the underlying preference relation and a welfare normalization for the best and the worst outcome. In the certainty additive representation, intertemporal risk aversion can be interpreted as a risk aversion with respect to welfare gains and losses. I have elaborated a close connection between the concept of intertemporal risk aversion and the disentanglement of atemporal risk aversion from intertemporal substitutability in the one-commodity setting. While the latter characterizations of preference become good-dependent in a multi-commodity setting, intertemporal risk aversion has been shown to specify a relation between the two, which is independent of the particular commodity under observation. Part II of my dissertation assumed a simplified two period setup with stationary preferences and a zero discount rate. The next chapter extends the concept of intertemporal risk aversion to the general multiperiod setting.

Part III

Extensions and Refinements of Part II

Chapter 8

Multiperiod Extension

8.1 Multiperiod Extension of the Representation

Part III of my dissertation further analyzes the concept of intertemporal risk aversion and its representation in a multiperiod framework. This chapter derives the most general framework. First section 8.1 works out the multiperiod representation of non-stationary preferences that comply in every period with the von Neumann-Morgenstern axioms for choice under uncertainty, and allow for additive separability over time on certain consumption paths. To this end I employ a recursive formulation of uncertainty developed and termed 'temporal lotteries' by Kreps & Porteus (1978). Subsequently, section 8.2 adapts the axiomatic characterization of intertemporal risk aversion to the derived representational framework. It defines the (period-specific) measures of relative and absolute intertemporal risk aversion and gives a respective condition for uniqueness. Later in part III, chapter 9 examines the representational consequence of different assumptions on stationarity. As there is no canonical way of imposing stationarity in a finite time framework, I offer two different axioms for stationarity of risk attitude. The first yields stationarity of the uncertainty aggregation rules, while the second is motivated mainly by the assumption that the mere passage of time should not change preferences. Finally chapter 10 analyzes the attitude of a decision maker with respect to the timing of uncertainty resolution. The latter is closely related to the measures of intertemporal risk aversion. In particular, I derive a representation for an intertemporally risk averse decision maker who is indifferent with respect to the timing of uncertainty resolution. This setting allows to reduce the temporal lottery setup, where uncertainty is expressed recursively over periods, to 'standard' lotteries, where uncertainty is expressed as a probability distribution over consumption paths. For a one commodity setting, this representation allows to disentangle atemporal risk aversion from intertemporal substitutability within an intertemporal expected utility model.

The first part of this chapter extends the representation of theorem 2 to an arbitrary finite planning horizon. Hereby I allow for general non-stationary preferences. The section is mainly technical. Instead of the more widespread framework of atemporal lotteries corresponding to probability measures over consumption paths, it involves the richer framework of temporal lotteries introduced by Kreps & Porteus (1978). The latter is, as Kreps & Porteus (1978) have shown, a natural extension of the classical von Neumann & Morgenstern (1944) setup to an intertemporal setting. It is also the standard framework in the literature disentangling atemporal risk aversion from intertemporal substitutability and thus a natural setting for a general definition of intertemporal risk aversion. When I introduced the time structure for the preceding chapters, I pointed out that within the general framework it only accounts for one and a half periods. Now, at the beginning of every period the decision maker faces uncertainty, not only over the future, but also over the outcome in the respective period. I identify period T with the last period. As before in T the decision maker has preferences over all lotteries on the space of outcomes $\tilde{X}_T \equiv X$, which are modeled as elements of $P_T \equiv \Delta(X)$. In the preceding chapters, choices in F corresponded to pairs of certain outcomes in period F and lotteries in period T, i.e. elements of the space $\tilde{X}_{T-1} \equiv X \times \Delta(X)$. In the general multiperiod framework, however, the second-last period $T-1$ starts before uncertainty over the respective period has resolved. Then, preferences in period $T-1$ are expressed by a relation $\succeq_{T-1}$ on the space of lotteries over $\tilde{X}_{T-1}$, which I denote by $P_{T-1} \equiv \Delta(\tilde{X}_{T-1}) = \Delta(X \times P_T)$. Note the recursive structure of the definition. The uncertainty at the beginning of period $T-1$ is not modeled as a probability distribution over the Cartesian product of outcomes in $T-1$ and T. Rather, it is defined as uncertainty over the outcome in $T-1$ and the lottery faced in period T. For a detailed introduction to recursive lotteries see also Kreps & Porteus (1978). Chapter 10.2 works out the relation between these recursive lotteries and probability measures that are defined directly on consumption paths.

In general, define $\tilde{X}_T = X$ and recursively $\tilde{X}_{t-1} = X \times \Delta(\tilde{X}_t)$ for all $t \in \{2, ..., T\}$. Equip the set of Borel probability measures on $\tilde{X}_t$, denoted by $P_t \equiv \Delta(\tilde{X}_t)$, with the Prohorov metric as to render $\tilde{X}_{t-1}$ compact in the product topology. The elements p_t of P_t are called (period t) lotteries. Preferences in period t are defined on the set P_t and denoted by $\succeq_t$ ($\subset P_t \times P_t$). The set of degenerate lotteries in P_t is identified with the set $\tilde{X}_t$ in the usual way. An uncertainty aggregation rule in period t is defined as a functional $\mathcal{M}^{f_t} : P_t \times \mathcal{C}^0(\tilde{X}_t) \rightarrow \mathrm{I\!R}$ with $\mathcal{M}^{f_t}(p_t, \tilde{u}_t) = f_t^{-1} \int_{\tilde{X}_t} dp_t \, f_t \circ \tilde{u}_t(\tilde{x}_t)$. To allow for general non-stationary preference I weaken the axiom of certainty additivity as follows:

A4' (certainty additivity) There exist functions $u_t \in \mathcal{C}^0(X)$, $t \in \{1, ..., T\}$, such that for all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^1$:

$$\mathsf{x} \succeq \mathsf{x}' \Leftrightarrow \sum_{t=1}^{T} u_t(\mathsf{x}_t) \geq \sum_{t=1}^{T} u_t(\mathsf{x}'_t). \tag{8.1}$$

Axiom 4' requires additive separability on certain consumption paths. However as opposed to axiom 4 the functions u_t are allowed to vary arbitrarily over time. In particular, tastes that are represented by a sequence $(u_t)_{t \in \{1,...,T\}}$ can even reverse between two periods. To define the set of Bernoulli utility functions for $\succeq_t$, the definition on page 77 just gains a time index, i.e. $B_{\succeq_t} = \{u_t \in \mathcal{C}^0(X) : [x]_t \succeq_t [x']_t \Leftrightarrow u_t(x) \geq u_t(x') \, \forall \, x, x' \in X\}$. Recall that axiom 4' only implies the *existence of a* certainty additive Bernoulli utility function. For other Bernoulli utility functions equation (8.1) generally does not hold.

The time consistency requirement is adapted to the set of preference relations $\succeq \equiv (\succeq_1, ..., \succeq_T)$ in the multiperiod framework as follows:

A5' (time consistency) For all $t \in \{1, ..., T\}$:

$$(x_t, p_{t+1}) \succeq_t (x_t, p'_{t+1}) \quad \Leftrightarrow \quad p_{t+1} \succeq_{t+1} p'_{t+1} \quad \forall \, x_t \in X, \; p_{t+1}, p'_{t+1} \in P_{t+1} \, .$$

The interpretation is equivalent to axiom A5. It is a requirement for choosing between two consumption plans in period t that yield a degenerate lottery with a coinciding entry in the respective period. For these choice situations, axiom 5' demands that in period t the decision maker shall prefer the plan that gives rise to the lottery that is preferred in period $t+1$.[1] The notation regarding the codomains of the functions u and g is adapted to the multiperiod setting by defining $\underline{U}_t = \min_{x \in X} u_t(x)$, $\overline{U}_t = \max_{x \in X} u_t(x)$ and $U_t = [\underline{U}_t, \overline{U}_t]$, as well as $\underline{G}_t = g_t(\underline{U}_t)$, $\overline{G}_t = g_t(\overline{U}_t)$, $G_t = [\underline{G}_t, \overline{G}_t]$ and $\Delta G_t = \overline{G}_t - \underline{G}_t$ for all $t \in \{1, ..., T\}$. Moreover let $\Gamma_t = (\underline{G}_t, \overline{G}_t)$. I assume for all following assertions that the decision maker faces at least two periods ($T \geq 2$) and that he is not indifferent between at least two outcomes in each period, i.e. for all $t \in \{1, ..., T\}$ there exist $x_t, x'_t \in X$ such that $[x]_t \succ_t [x']_t$. Then, the following representation holds.

Theorem 4: Let there be given a sequence of preference relations $(\succeq_t)_{t \in \{1,...,T\}}$ on $(P_t)_{t \in \{1,...,T\}}$ and a sequence of Bernoulli utility functions $(u_t)_{t \in \{1,...,T\}}$ with $u_t \in$

[1] This is *time consistency* in the sense of Kreps & Porteus (1978). The only difference to axiom 5 is that (x_t, p_{t+1}) now has the interpretation of a degenerate lottery.

$B_{\succeq_t}$. The sequence of preference relations $(\succeq_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting)

ii) A4' for $\succeq_1|_{\mathbf{X}^1}$ (certainty additivity)

iii) A5' (time consistency)

if and only if, for all $t \in \{1, ..., T\}$ there exist strictly increasing[2] and continuous functions $f_t : U_t \to \mathbb{R}$ and $g_t : U_t \to \mathbb{R}$ such that with defining

v) the normalization constants $\theta_T = 1$, $\vartheta_T = 0$ and for $t < T$

$$\theta_t = \frac{\Delta G_t}{\sum_{\tau=t}^{T} \Delta G_\tau} \quad \text{and} \quad \vartheta_t = \frac{\overline{G}_{t+1}\underline{G}_t - \underline{G}_{t+1}\overline{G}_t}{\Delta G_t} \quad \text{and}$$

vi) recursively the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ by $\tilde{u}_T(x_T) = u(x_T)$ and

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = g_{t-1}^{-1}\left[\theta_{t-1}\, g_{t-1} \circ u_{t-1}(x_{t-1}) + \frac{\theta_{t-1}}{\theta_t}\, g_t \circ \mathcal{M}^{f_t}(p_t, \tilde{u}_t) + \frac{\theta_{t-1}}{\theta_t}\vartheta_{t-1}\right]$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t\,. \tag{8.2}$$

Moreover, $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ and $(u_t, f'_t, g'_t)_{t\in\{1,...,T\}}$ both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, for all $t \in \{1, ..., T\}$ there exist constants $a_t^f \in \mathbb{R}_{++}$ and $b_t^f \in \mathbb{R}$ such that $f_t = a_t^f f'_t + b_t^f$, as well as constants $a^g \in \mathbb{R}_{++}$ and $b_t^g \in \mathbb{R}$ such that $g_t = a^g g'_t + b_t^g$.

A sequence of triples $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ as above is called a representation in the sense of theorem 4 for the set of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$. The representation theorem recursively constructs an aggregate utility $\tilde{u}_t$ that depends on the utility in the respective period $u_t(x_t)$, as well as the aggregate utility derived from a particular lottery p_{t+1} over the future. The construction of aggregate utility features for $t \in \{1, ..., T-1\}$ the intertemporal aggregation rule

$$\begin{aligned}
\mathcal{N}^{g_t,g_{t+1}} \quad &: \; U_t \times U_{t+1} \to \mathbb{R} \\
\mathcal{N}^{g_t,g_{t+1}}(\cdot,\cdot) &= g_t^{-1}\left[\theta_t g_t(\cdot) + \theta_t\theta_{t+1}^{-1} g_{t+1}(\cdot) + \theta_t\theta_{t+1}^{-1}\vartheta_t\right] \\
&= g_t^{-1}\left[\theta_t\, g_t(\cdot) + (1-\theta_t)\frac{\Delta G_t}{\Delta G_{t+1}}\big(g_{t+1}(\cdot) + \vartheta_t\big)\right]\,.
\end{aligned}$$

The reformulation in the last line is an immediate consequence of the relation $\theta_t\theta_{t+1}^{-1} = (1-\theta_t)\frac{\Delta G_t}{\Delta G_{t+1}}$.[3] The normalization constants ensure that the domain of g_t^{-1} in the in-

[2]Alternatively the theorem can be stated replacing increasing by monotonic for $(f_t)_{t\in\{1,...,T\}}$ and demanding that either all $(g_t)_{t\in\{1,...,T\}}$ are strictly increasing or that all are strictly decreasing.

[3]See equation C.2 in the appendix on page 209.

tertemporal aggregation rule $\mathcal{N}^{g_t,g_{t+1}}$ coincides with G_t in order to make the inverse well defined. For stationary preferences chapter 9 will show that $g_t = g_{t'}$ and $\vartheta_t = 0$ for all $t, t' \in \{1, ..., T\}$, giving rise to the formulation employed in part II. As opposed to the respective representation in theorem 2, however, in the general stationary setting the normalization constants θ_t give rise to discounting. Whenever the outcome in the respective period or the lottery faced in the future are not known with certainty, the uncertainty aggregation rule $\mathcal{M}^{f_t}$ is applied in order to return the certainty equivalent overall welfare. As formulated in equation (8.2), the resulting expression $\mathcal{M}^{f_t}(p_t, \tilde{u}_t)$ evaluates period t lotteries in a way representing choice between general uncertain consumption or outcome plans faced in t. In this recursive evaluation, the representation of theorem 4 not only features period-specific intertemporal aggregation rules, but also period-specific uncertainty aggregation rules. Chapter 9 analyzes two different assumptions on risk stationarity that relate uncertainty aggregation in different periods.

With respect to the moreover part, observe that the functions $(f_t)_{t\in\{1,...,T\}}$ are determined only up to individual affine transformations, while for the functions $(g_t)_{t\in\{1,...,T\}}$ the same multiplicative constant has to apply for all periods. In order to condense the statement of the moreover part in a way that proves particularly useful in the gauge corollaries, I introduce the following notation. For $a \in \mathbb{R}_{++}$ define the set of affine transformations A^a with elements $\mathfrak{a}^a$ by $\mathsf{A}^a = \{\mathfrak{a}^a : \mathbb{R} \to \mathbb{R} : \mathfrak{a}^a(z) = az + b, b \in \mathbb{R}\}$. In this notation the moreover part of theorem 4 writes as: Moreover, $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ and $(u_t, f'_t, g'_t)_{t\in\{1,...,T\}}$ both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, there exists $a \in \mathbb{R}_{++}$ such that for all $t \in \{1, ..., T\}$ there exist $\mathfrak{a}^+_t \in \mathsf{A}^+$ and $\mathfrak{a}^a_t \in \mathsf{A}^a$ such that $(g'_t, f'_t) = (\mathfrak{a}^+_t f_t, \mathfrak{a}^a_t g_t)$. Let me point out that the functions $(g_t)_{t\in\{1,...,T\}}$, characterizing intertemporal aggregation, are determined already by the preferences over certain consumption paths, i.e. by $\succeq_1 |_{\mathsf{X}^1}$. Moreover, comparing only certain consumption paths, the normalization parameter ϑ_t serves exclusively to allow for the renormalization g_t^{-1}. It has no representational function in the sense that the representation

$$g_t^{-1}\left[\theta_t g_t \circ u(x_t) + \theta_t\theta_{t+1}^{-1} g_{t+1} \circ \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) + \theta_t\theta_{t+1}^{-1}\vartheta_t\right]$$

is a strictly increasing transformation of the expression

$$\theta_t g_t \circ u(x_t) + \theta_t\theta_{t+1}^{-1} g_{t+1} \circ \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \ ,$$

which does not depend on ϑ_t. However, as soon as uncertainty is introduced, the representation becomes

$$f_t^{-1}\left\{\int dp_t^{x_t,p_{t+1}} f_t \circ g_t^{-1}\left[\theta_t g_t \circ u(x_t) + \theta_t\theta_{t+1}^{-1} g_{t+1} \circ \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) + \theta_t\theta_{t+1}^{-1}\vartheta_t\right]\right\}. \quad (8.3)$$

In equation (8.3) no strictly increasing transformation can eliminate ϑ_t for nondegenerate lotteries. Thus, the normalization constant gains representational responsibility. However, it is possible to eliminate ϑ_t from the representation by adding an additional requirement for the representing functions g_t. The moreover part of theorem 4 admits individual affine translations for the representing sequence $(g_t)_{t\in\{1,...,T\}}$. This individual freedom can be used to yield the following modification of theorem 4.

Theorem 5: Let there be given a sequence of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ and a sequence of Bernoulli utility functions $(u_t)_{t\in\{1,...,T\}}$ with $u_t \in B_{\succeq_t}$. The sequence of preference relations $(\succeq_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1,...,T\}$ (vNM setting)

ii) A4' for $\succeq_1|_{\mathbf{X}^1}$ (certainty additivity)

iii) A5' (time consistency)

if and only if, for all $t \in \{1,...,T\}$ there exist strictly increasing and continuous functions $f_t : U_t \to \mathbb{R}$ and $g_t : U_t \to \mathbb{R}$, the latter satisfying $\frac{\overline{G}_{t+1}}{\underline{G}_{t+1}} = \frac{\overline{G}_t}{\underline{G}_t}$, such that with defining

v) the normalization constants

$$\theta_t = \frac{\Delta G_t}{\sum_{\tau=t}^{T} \Delta G_\tau} \quad \text{for } t \in \{1,..,T\} \text{ and}$$

vi) recursively the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ by $\tilde{u}_T(x_T) = u(x_T)$ and

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = g_{t-1}^{-1}\left[\theta_{t-1}\, g_{t-1} \circ u_{t-1}(x_{t-1}) + \theta_{t-1}\theta_t^{-1}\, g_t \circ \mathcal{M}^{f_t}(p_t, \tilde{u}_t)\right]$$

it holds for all $t \in \{1,...,T\}$ that

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t\, .$$

Moreover, $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ and $(u_t, f'_t, g'_t)_{t\in\{1,...,T\}}$ both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, for all $t \in \{1,...,T\}$ there exist constants $a_t \in \mathbb{R}_{++}$ and $b_t \in \mathbb{R}$ such that $f_t = a_t f'_t + b_t$, as well as constants $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $g_t = a g'_t + b\Delta G_t$.

The only difference between the two representations is that the additional requirement $\frac{\overline{G}_{t+1}}{\underline{G}_{t+1}} = \frac{\overline{G}_t}{\underline{G}_t}$ picks a particular sequence of representations $(g_t)_{t\in\{1,...,T\}}$ such that the intertemporal aggregation rule in theorem 5,

$$\mathcal{N}_*^{g_t,g_{t+1}} \quad : \quad U_t \times U_{t+1} \to \mathbb{R}$$

$$\mathcal{N}_*^{g_t,g_{t+1}}(\cdot,\cdot) = g_t^{-1}\left[\theta_t g_t(\cdot) + \theta_t\theta_{t+1}^{-1} g_{t+1}(\cdot)\right]\,,$$

spares the normalization constants ϑ_t. A consequence of the additional requirement for the functions g_t is that they are no longer free up to individual translation parameters b_t.

Observe that this additional determinateness of the sequence $(g_t)_{t\in\{1,...,T\}}$ is no longer due to information that is conveyed already by preference over certain consumption paths. The information stems from absorbing the normalization constant ϑ_t, which was seen to gain representational relevance only for the choice between nondegenerate lotteries. As theorem 5 breaks the symmetry in the freedom to choose g_t, f_t and $u_t \in B_{\succeq_t}$, the gauge corollaries are stated in terms of representation theorem 4, allowing for arbitrary choices of g_t. Gauging works out as in chapter 6.4. The only difference is that in the more general framework developed above, I can choose the Bernoulli utility functions independently for every period and, therefore, gauge the functions g_t or f_t differently for different periods. The gauge corollaries build on the following analogon to lemma 1 in chapter 6.

Lemma 4: Let the triple (u_τ, f_τ, g_τ) be part of a representation $(u_t, f_t, g_t)_{t\in\{1,...,\tau,...,T\}}$ for $(\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 4. Moreover, let $s_\tau : U_\tau \to \mathrm{I\!R}$ be a strictly increasing and continuous transformation and define the triple $(u'_\tau, f'_\tau, g'_\tau) = (s_\tau \circ u_\tau, f_\tau \circ s_\tau^{-1}, g_\tau \circ s_\tau^{-1})$. Then, letting $(u'_t, f'_t, g'_t) = (u_t, f_t, g_t)\,\forall\, t \neq \tau$, the sequence $(u'_t, f'_t, g'_t)_{t\in\{1,...,T\}}$ is a representation of $(\succeq_t)_{t\in\{1,...,T\}}$.

Using lemma 4 I can always choose the Bernoulli utility functions in a way to render any desired form of the uncertainty aggregation rule. This is stated in

Corollary 6 (f-gauge) :

For any sequence of strictly increasing and continuous functions $\boldsymbol{f}_t : \mathrm{I\!R} \to \mathrm{I\!R}$, $t \in \{1, ..., T\}$, the following equivalence holds:

The sequence of preference relations $(\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

$i - iii)$ of theorem 4,

if and only if, for all $t \in \{1, ..., T\}$ there exist continuous functions $\boldsymbol{u}_t : X \to \mathrm{I\!R}$ as well as strictly increasing and continuous functions $\boldsymbol{g}_t : U_t \to \mathrm{I\!R}$ such that with defining

$v - vi)$ of theorem 4,

the representation (8.2) of theorem 4 holds for all $t \in \{1, ..., T\}$.

Moreover, $(u_t, g_t)_{t\in\{1,...,T\}}$ and $(u'_t, g'_t)_{t\in\{1,...,T\}}$ both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, there exists $a \in \mathrm{I\!R}_{++}$ such that for all $t \in \{1, ..., T\}$ there are affine transformations $\mathsf{a}_t^+ \in \mathsf{A}^+$ and $\mathsf{a}_t^a \in \mathsf{A}^a$ such that $(u'_t, g'_t) = (f_t^{-1}\mathsf{a}_t^+ f_t\, u_t\,,\ \mathsf{a}_t^a\, g_t\, f_t^{-1}\mathsf{a}_t^{+^{-1}} f_t)$.

In particular, choosing all functions f_t as the identity, corollary 6 yields the *Kreps Porteus gauge*. In that case, uncertainty aggregation is always described by the expected value operator. In general, however, this choice comes at the cost of a non-linear aggregation of Bernoulli utility over time. But as discussed in chapter 7.3, a linear aggregation of Bernoulli utility over time can be desirable, e.g. for an interpretation of Bernoulli utility in terms of welfare. The following lemma serves as the basis for such a certainty additive gauge.

Corollary 7 (g-gauge) :

For any sequence of strictly increasing and continuous functions $\boldsymbol{g}_t : \mathbb{R} \rightarrow \mathbb{R}$, $t \in \{1, ..., T\}$, the following equivalence holds:

The sequence of preference relations $(\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

$i - iii)$ of theorem 4,

if and only if, for all $t \in \{1, ..., T\}$ there exist continuous functions $\boldsymbol{u}_t : X \rightarrow \mathbb{R}$ as well as strictly increasing and continuous functions $\boldsymbol{f}_t : U_t \rightarrow \mathbb{R}$ such that with defining

$v - vi)$ of theorem 4,

the representation (8.2) of theorem 4 holds for all $t \in \{1, ..., T\}$.

Moreover, $(u_t, g_t)_{t\in\{1,...,T\}}$ and $(u'_t, g'_t)_{t\in\{1,...,T\}}$ both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, there exists $a \in \mathbb{R}_{++}$ such that for all $t \in \{1, ..., T\}$ there are affine transformations $\mathfrak{a}_t^+ \in \mathsf{A}^+$ and $\mathfrak{a}_t^a \in \mathsf{A}^a$ such that $(u'_t, f'_t) = (g_t^{-1} \mathfrak{a}_t^a g_t \; u_t \, , \; \mathfrak{a}_t^+ f_t \; g_t^{-1} \mathfrak{a}_t^{a-1} \; g_t)$.

In particular, choosing the functions g_t as the identity for all $t \in \{1, ..., T\}$ yields the *certainty additive gauge*.[4] Here, Bernoulli utility is time-additive over certain consumption paths. For a stationary setting without discounting, chapter 7.3 has explained how in such a representation the concept of intertemporal risk aversion can be interpreted as risk aversion on welfare. The next section extends the concept of intertemporal risk aversion into the general framework derived above.

8.2 Intertemporal Risk Aversion

In this section I introduce the general, axiomatic characterization of intertemporal risk aversion. In the special case of a decision maker with stationary preference and a zero rate of pure time preference, the definition is equivalent to the form introduced in

[4]Note that, for the definition of the normalization constants θ_t, this is a special case where $\Delta G_t = \Delta U_t$.

chapter 7.2. To tie up with the discussion in part II, I first state the characterization for a two period setting. Subsequently, two alternative extensions of the axiom to an arbitrary number of periods are offered, both of which yield the same characterization in terms of the representation derived in the preceding section. The main difference in the formulation with regard to chapter 7.2 is the following. Relaxing the requirements of a zero rate of time preference and stationarity imply that the definition has to consider lotteries over general consumption paths, rather then over individual outcomes in a single period.[5]

In the two period setting, a decision maker is said to exhibit *weak intertemporal risk aversion* (in period 1), if and only if, the following axiom is satisfied:

A6$^{\mathrm{w}}_{\mathrm{tp}}$ (weak intertemporal risk aversion) For all $x_1, x_1', x_2, x_2' \in X$ holds

$$(x_1, x_2) \sim_1 (x_1', x_2') \quad \Rightarrow \quad (x_1, x_2) \;\; \succeq_1 \;\; \frac{1}{2}(x_1, x_2') + \frac{1}{2}(x_1', x_2). \tag{8.4}$$

The index 'w' at the axiom's label abbreviates again 'weak' as opposed to 's' for 'strict', while 'tp' denotes the particular two period setting. A decision maker is said to exhibit *strict intertemporal risk aversion* (in period 1), if and only if, the following axiom is satisfied:

A6$^{\mathrm{s}}_{\mathrm{tp}}$ (strict intertemporal risk aversion) For all $x_1, x_1', x_2, x_2' \in X$ holds

$$(x_1, x_2) \sim_1 (x_1', x_2') \;\wedge\; x_2 \not\sim_2 x_2' \quad \Rightarrow \quad (x_1, x_2) \;\; \succ_1 \;\; \frac{1}{2}(x_1, x_2') + \frac{1}{2}(x_1', x_2). \tag{8.5}$$

I start with the interpretation of the strict axiom. The first part of the premise in axiom A6$^{\mathrm{s}}_{\mathrm{tp}}$ states that a decision maker is indifferent between a certain consumption path delivering outcome x_1 in the first period and outcome x_2 in the second period, and another certain consumption path delivering outcomes x_1' and x_2' respectively. The second part of the premise requires that the decision maker values either x_2 higher than x_2' or vice versa. Let me assume he prefers outcome x_2' over outcome x_2 in the second period. In combination with the first part of the premise, this preference implies that the decision maker values x_1 higher than x_1' in the first period.[6] Moreover, the indiffer-

[5]Precisely, admitting a non-zero rate of time preference implies the necessity to consider lotteries over paths rather than over single period outcomes. Allowing for non-stationary preference results in the additional need to include non-constant certain consumption paths in the axiom's comparisons. Chapter 9.4 gives a simplified version of the axiom for the case of general *stationary* preference, keeping consumption paths on the left hand side of the comparisons in equation (8.4) constant over time.

[6]Again, the interpretation assumes that the axioms of choice of the last section, as gathered in

ence with respect to the intertemporal trade-off in the first part of the premise implies that there is some equivalence between the superiority of x_1 over x'_1 and the inferiority of x_2 with respect to x'_2. To arrive at the right hand side of equation (8.5), reassemble the outcomes to the new consumption paths (x_1, x'_2) and (x'_1, x_2). As opposed to the consumption path (x_1, x_2), the consumption path (x_1, x'_2) features a higher consumption in the second period, while first period consumption coincides. On the other hand, the consumption path (x'_1, x_2) delivers an inferior outcome in the first period, but consumption in the second is the same as in (x_1, x_2). Moreover, in the sense of indifference in the intertemporal trade-off, there is an evaluative equivalence between the superiority of the consumption path (x_1, x'_2) with respect to (x_1, x_2), and the inferiority of the consumption path (x'_1, x_2) with respect to (x_1, x_2). Now, the right hand side of equation (8.5) offers the choice between the consumption path (x_1, x_2) with certainty and a lottery that yields, with equal probability, either the superior consumption path (x'_1, x_2) or the inferior consumption path (x_1, x'_2). Then, the axiom requires an intertemporally risk averse decision maker to prefer the certain consumption path over the lottery. Again, the intuition is that he might be better off in the lottery than with the certain consumption path (x_1, x_2), but he might as well be worse off with equal probability. This differs from the decision problem in the premise, where the decision maker can be certain to get the higher outcome in the second period, if he decides for the inferior outcome in the first.

The interpretation for the *weaker* axiom $A6^{w}_{tp}$ is analogous. The only difference is that the implication of axiom $A6^{w}_{tp}$ just requires the lottery not to be strictly preferred to the certain outcome path. Therefore, the premise can allow that both consumption paths, (x_1, x_2) and (x'_1, x'_2), are evaluated identical with respect to their individual outcomes. If axiom $A6^{s}_{tp}$ ($A6^{w}_{tp}$) is satisfied with $\succ_1$ ($\succeq_1$) replaced by $\prec_1$ ($\preceq_1$), the decision maker is called a strong (weak) *intertemporal risk seeker*. If his preferences satisfy weak intertemporal risk aversion as well as weak intertemporal risk seeking, the decision maker is called *intertemporally risk neutral*.

Before I extend the axioms to an arbitrary planning horizon, let me revisit the example studied in chapter 7.4. An employee was offered a vacation lottery by his boss. The corresponding coin flip would have either extended his summer vacation from two to three weeks, or would have cut the vacation short by one week. As the employee declined the lottery, I wondered whether this rejection was due to risk aversion or whether it

theorem 5, are satisfied. Preferring x_1 over x'_1 in the first period corresponds formally to $[x_1] \succ_1 [x'_1]$. In terms of representation theorem 5, this relation implies that any Bernoulli utility function renders a higher value for outcome x_1 than for outcome x'_1.

was caused by a higher valuation for a second week of vacation than for a third week of vacation. Therefore, I asked the employee, whether he was willing to trade in a week of this year's summer vacation for an extra week of next year's summer vacation. Acknowledging that I observed a rather particular employee, who featured a stationary preference relation and a zero rate of time preference, this combined information was enough to decide whether he declined due to (intertemporal) risk aversion or due to judging a third week of vacation inferior to a second week of vacation. Now let me extend the example to those employees, who have general non-stationary preferences, by applying axiom A6^{s}_{tp} to the above setting. Hence, in order to find out whether such an employee is intertemporally risk averse, I have to offer him a different lottery than in chapter 7.4. Now, the coin flip should promise him either two weeks of vacation this year and three in the next, or, if the other side comes up, one week of vacation this year and two in the next. If he declines such an offer, but prefers to trade in one week of vacation this year for an extra week next year, the employee is known to be intertemporally risk averse.[7] If he accepts the lottery, but declines the certain trade, the employee is known to be an intertemporal risk seeker. If he declines both, the lottery and the certain trade, I only know that he is not, at the same time, intertemporally risk seeking and values the third week higher than the second. If he accepts both, I only know that he is not intertemporally risk averse and, at the same time, judges the third week inferior to the second. In these cases, I have to change the vacation payoffs, corresponding to a variation of the outcome paths $\mathsf{x} = (x_1, x_2)$ and $\mathsf{x}' = (x'_1, x'_2)$ in axiom A6^{s}_{tp}, in order to derive further information on the employees preference and risk attitude.

To state the axiom for an arbitrary finite planning horizon T, I need to introduce some notation. Given two consumption paths $\mathsf{x} = (x_t, x_{t+1}, ..., x_T) \in \mathsf{X}^t$ and $\mathsf{x}' = (x'_t, x'_{t+1}, ..., x'_T) \in \mathsf{X}^t$, I define the reassembled consumption path $(\mathsf{x}_{-i}, \mathsf{x}'_i) = (\mathsf{x}_t, ..., \mathsf{x}_{i-1}, \mathsf{x}'_i, \mathsf{x}_{i+1}, ..., x_T) \in \mathsf{X}^t$, as the consumption path that coincides with x in all but the i^{th} period, and in period i it coincides with the period i outcome of path x'. Note that for $i \in \{t, ..., T\}$ there are $T - t + 1$ different consumption paths $(\mathsf{x}_{-i}, \mathsf{x}'_i)$, each of length $T - t + 1$. Then, $\sum_{i=t}^{T} \frac{1}{T-t+1} (\mathsf{x}_{-i}, \mathsf{x}'_i)$ denotes a lottery that yields with equal probability any of these consumption paths $(\mathsf{x}_{-i}, \mathsf{x}'_i)$. This lottery can also be described as follows. Think about constructing a new consumption path out of the consumption path x, by keeping all but one of its entry. The entry that is changed, replaces the outcome x_i by the outcome x'_i. Now, the lottery draws with equal probability the period

[7] More precisely, he is known to be intertemporally risk averse for that particular welfare level. However, see chapter 10 to find (axiomatic) reasons, why a decision maker might be only either *everywhere* intertemporally risk averse or *everywhere* intertemporally risk seeking.

in which the consumption is exchanged.

A decision maker is said to exhibit *weak intertemporal risk aversion* in period $t < T$, if and only if the following axiom is satisfied:

A6 $^{\mathrm{w}}$ (weak intertemporal risk aversion) For all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^t$ holds

$$\mathsf{x} \sim_t \mathsf{x}' \quad \Rightarrow \quad \mathsf{x} \ \succeq_t \ \sum_{i=t}^{T} \tfrac{1}{T-t+1} \, (\mathsf{x}_{-i}, \mathsf{x}'_i).$$

A decision maker is said to exhibit *strict intertemporal risk aversion* in period $t < T$, if and only if the following axiom is satisfied:

A6 $^{\mathrm{s}}$ (strict intertemporal risk aversion) For all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^t$ holds

$$\mathsf{x} \sim_t \mathsf{x}' \quad \wedge \quad \exists\, \tau \in \{t, ..., T\} \text{ s.th. } [\mathsf{x}_\tau]_\tau \ \not\sim_\tau \ [\mathsf{x}'_\tau]_\tau$$

$$\Rightarrow \quad \mathsf{x} \ \succ_t \ \sum_{i=t}^{T} \tfrac{1}{T-t+1} \, (\mathsf{x}_{-i}, \mathsf{x}'_i).$$

I give an interpretation for the strict version of the axiom, which illustrates also the interpretation of its weak form. The premise, i.e. the first line of axiom A6 $^{\mathrm{s}}$, is analogous to that of axiom $\mathrm{A6}^{\mathrm{s}}_{\mathrm{tp}}$. The first part states that a decision maker is indifferent between the certain consumption paths x and x'. The second part of the premise requires that there exists at least one period, in which the decision maker is not indifferent between the outcome delivered in the respective period by consumption path x and the one delivered by consumption path x'.[8] Without loss of generality assume that there exists $\tau \in \{t, ..., T\}$ such that outcome x_τ is strictly preferred to outcome x'_τ (i.e. $[\mathsf{x}_\tau]_\tau \succ_\tau [\mathsf{x}'_\tau]_\tau$). Then[9], by the first part of the premise, there also has to exist a period τ', in which the outcome $x_{\tau'}$ is judged inferior to the outcome $x'_{\tau'}$. Therefore, the second part of the premise implies that there exists a consumption path $(\mathsf{x}_{-\tau'}, \mathsf{x}'_{\tau'})$ that is judged superior to the consumption path x as well as a consumption path $(\mathsf{x}_{-\tau}, \mathsf{x}'_\tau)$ that is judged inferior to x. Of course, there can be several consumption paths of type $(\mathsf{x}_{-i}, \mathsf{x}'_i)$ with $i \in \{t, ..., T\}$ that are judged superior or inferior with respect to the consumption path x. However, the outcomes x'_i that make the paths $(\mathsf{x}_{-i}, \mathsf{x}'_i)$ superior and those that make the paths inferior with respect to x, balance each other in the sense of the intertemporal trade-off given in the first part of the premise. Then, the second line of axiom A6 $^{\mathrm{s}}$ demands that for consumption satisfying the above conditions, an intertemporally risk averse decision maker should prefer the consumption path x with certainty over the lottery that yields with equal probability any of the consumption paths $(\mathsf{x}_{-i}, \mathsf{x}'_i)$, some

[8]Recall the definition of $[x_t]_t$ as the consumption path $(x_t, x^0, ..., x^0)$ that yields outcome x_t in period t and some arbitrary, but commonly fixed, baseline consumption $x^0 \in X$ from period $t+1$ on.

[9]Assuming that the axioms of choice given in the preceding section prevail.

of which make him better off and some of which make him worse off.

The interpretation for the *weaker* axiom A6$^{\text{w}}$ is analogous, except that the consumption path x is allowed to coincide with x', and the implication only requires that the lottery is not strictly preferred to the certain consumption path. If axiom A6$^{\text{s}}$ (A6$^{\text{w}}$) is satisfied with $\succ_t$ ($\succeq_t$) replaced by $\prec_t$ ($\preceq_t$) the decision maker is called a strong (weak) *intertemporal risk seeker*. If a decision maker's preferences satisfy weak intertemporal risk aversion as well as weak intertemporal risk seeking, the decision maker is called *intertemporally risk neutral*.

Before stating the theorem that characterizes intertemporal risk aversion in terms of the representation of theorem 4, I want to offer an alternative axiomatic characterization of intertemporal risk aversion in the multiperiod setting, which only involves a lottery over *two* consumption paths. These consumption paths are constructed by separating the relatively better outcomes of x with respect to x' from the relatively worse outcomes of x with respect to x'. Precisely, define for $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^t$ the consumption paths $x^{\text{high}}(\mathsf{x}, \mathsf{x}')$ and $x^{\text{low}}(\mathsf{x}, \mathsf{x}')$ in X^t by

$$(x^{\text{high}}(\mathsf{x}, \mathsf{x}'))_\tau = \begin{cases} \mathsf{x}'_\tau \text{ if } [\mathsf{x}'_\tau]_\tau \succ_\tau [\mathsf{x}_\tau]_\tau \\ \mathsf{x}_\tau \text{ if } [\mathsf{x}_\tau]_\tau \succeq_\tau [\mathsf{x}'_\tau]_\tau \end{cases}$$

and

$$(x^{\text{low}}(\mathsf{x}, \mathsf{x}'))_\tau = \begin{cases} \mathsf{x}'_\tau \text{ if } [\mathsf{x}_\tau]_\tau \succeq_\tau [\mathsf{x}'_\tau]_\tau \\ \mathsf{x}_\tau \text{ if } [\mathsf{x}'_\tau]_\tau \succ_\tau [\mathsf{x}_\tau]_\tau \end{cases}$$

for $\tau \in \{t, ..., T\}$. The consumption path $x^{\text{high}}(\mathsf{x}, \mathsf{x}')$ collects the better outcomes of every period, while $x^{\text{low}}(\mathsf{x}, \mathsf{x}')$ collects the inferior outcomes of every period. In this notation the definition of *weak intertemporal risk aversion* in period $t < T$ can also be stated as follows:

A6$^{\text{w}}_*$ (weak intertemporal risk aversion) For all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^t$ holds

$$\mathsf{x} \sim_t \mathsf{x}' \quad \Rightarrow \quad \mathsf{x} \succeq_t \tfrac{1}{2}\, x^{\text{high}}(\mathsf{x}, \mathsf{x}') + \tfrac{1}{2}\, x^{\text{low}}(\mathsf{x}, \mathsf{x}').$$

And *strict intertemporal risk aversion* in period $t < T$ can be written as:

A6$^{\text{s}}_*$ (strict intertemporal risk aversion) For all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^t$ holds

$$\mathsf{x} \sim_t \mathsf{x}' \quad \wedge \quad \exists\, \tau \in \{t, ..., T\} \text{ s.th. } [\mathsf{x}_\tau]_\tau \not\sim_\tau [\mathsf{x}'_\tau]_\tau$$

$$\Rightarrow \quad \mathsf{x} \succ_t \tfrac{1}{2}\, x^{\text{high}}(\mathsf{x}, \mathsf{x}') + \tfrac{1}{2}\, x^{\text{low}}(\mathsf{x}, \mathsf{x}').$$

The interpretations are analogous to those of axioms A6$^{\text{w}}$ and A6$^{\text{s}}$. However, the 'worse off' versus 'better off' trade-off in the lottery can be observed more directly. Moreover, for long time horizons, the formulation in axioms A6$^{\mathrm{w}}_{*}$ and A6$^{\mathrm{s}}_{*}$ reduces the outcome paths offered by the lottery significantly. Note that for the case of two periods, both axioms A6$^{\mathrm{w}}$ and A6$^{\mathrm{w}}_{*}$ coincide with axiom A6$^{\mathrm{w}}_{\mathrm{tp}}$, and both axioms A6$^{\mathrm{s}}$ and A6$^{\mathrm{s}}_{*}$ are identical to axiom A6$^{\mathrm{s}}_{\mathrm{tp}}$. Theorem 6 shows the equivalence of the two formulations within a preference setup as described in the preceding section. Most importantly however, it translates the axiomatic characterization of intertemporal risk aversion into a functional characterization for representations in the sense of theorem 4.

Theorem 6: Let the sequence of triples $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ represent the preferences $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 5. Furthermore let $t \in \{1, ..., T-1\}$. The following assertions hold:

a) A decision maker is strictly intertemporally risk averse [seeking] in period t in the sense of axiom A6$^{\mathrm{s}}$, if and only if, $f_t \circ g_t^{-1}(z)$ is strictly concave [convex] in $z \in \Gamma_t$.

b) A decision maker is weakly intertemporally risk averse [seeking] in period t in the sense of axiom A6$^{\mathrm{s}}$, if and only if, $f_t \circ g_t^{-1}(z)$ is concave [convex] in $z \in \Gamma_t$.

c) A decision maker is intertemporally risk neutral in period t, if and only if, $f_t \circ g_t^{-1}(z)$ is linear in $z \in \Gamma_t$.

d) The above assertions hold as well, if axiom A6$^{\mathrm{s}}$ is replaced by axiom A6$^{\mathrm{s}}_{*}$, and if axiom A6$^{\mathrm{w}}$ is replaced by axiom A6$^{\mathrm{w}}_{*}$.

In theorem 6 intertemporal risk aversion is characterized by the functions $f_t \circ g_t^{-1}$. Note, that for period T, the term $f_T \circ g_T^{-1}$ is determined by the underlying preferences $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ to the same degree as the compositions $f_t \circ g_t^{-1}$ for any other period. Therefore, theorem 6 can be used to extend the definition of intertemporal risk aversion to the last period of the planning horizon.[10] The interpretation of the assertions is the same as for theorem 3. For a detailed discussion I refer to chapter 7. However, the above extension to the general multiperiod setting allows the functions characterizing intertemporal risk aversion to change arbitrarily over time. In the one-commodity Epstein Zin gauge, theorem 6 shows that intertemporal risk aversion goes along with standard risk aversion dominating a decision makers aversion to substitute consumption over time. Considering the certainty additive gauge ($g = \mathrm{id}$), assertion a) states that intertemporal risk aversion is equivalent to a strictly concave function f. As this gauge was implicitly applied in the setup of sections 5.2 and 5.3, where a strictly concave f has been identified to characterize precautionary behavior (compare page 69), strict

[10] Of course, this is only possible because the sequence of preference relations $\succeq$ is defined over at least two periods. Otherwise, i.e. in the one-period or atemporal case, g_T would not be determined at all.

intertemporal risk aversion corresponds again to a precautionary evaluation of a threat of harm scenario.

As discussed in chapter 7.3, in the $g = \text{id}$-gauge, certainty additive Bernoulli utility is given the interpretation of welfare u^{welf}. With such a notion of welfare, intertemporal risk aversion can be interpreted as risk aversion with respect to welfare gains and losses. Just as in chapter 7.3, I can also reinterpret the axioms characterizing intertemporal risk aversion in this section in terms of risk aversion on welfare. For example axiom A6$^{\text{s}}_{*}$ gains the following interpretation. The first part of the premise requires that for two consumption paths, x and x', the per period welfare adds up to the same overall welfare. The second part of the premise requires that at least in one period, the welfare gained from consumption path x differs from that gained from consumption path x'. The consumption path $x^{\text{high}}(\mathrm{x}, \mathrm{x}')$ collects for every period the outcome x_t or x'_t that renders the comparatively higher welfare, while the consumption path $x^{\text{low}}(\mathrm{x}, \mathrm{x}')$ collects the outcome x_t or x'_t that yields the comparatively lower welfare. Then, the lottery between these 'high welfare' and 'low welfare' consumption paths constructed in axiom A6$^{\text{s}}_{*}$ renders in expectation the same welfare as the certain consumption path x. A decision maker who is strictly risk averse on welfare, i.e. strictly intertemporally risk averse, is defined by preferring the certain consumption path x over the welfare lottery that leaves him with equal probability either worse or better off, and yields the same welfare as the certain consumption path in expectation.

For a quantitative measurement of risk aversion, the two definitions of chapter 7.4 gain a time index. For a twice differentiable function $f_t \circ g_t^{-1} : \Gamma_t \to \mathbb{R}$ define a measure of *relative intertemporal risk aversion in period* t as the function:

$$\text{RIRA}_t : \Gamma_t \to \mathbb{R}$$

$$\text{RIRA}_t(z) = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)}\, z\,. \tag{8.6}$$

Analogously define the coefficient of *absolute intertemporal risk aversion in period* t as the function:

$$\text{AIRA}_t : \Gamma_t \to \mathbb{R}$$

$$\text{AIRA}_t(z) = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)}\,. \tag{8.7}$$

To make these quantities well defined, the affine freedom in the representation corresponding to the evaluation of certain consumption paths has to be fixed, just as in chapter 7.4. In a general gauge, this evaluation is characterized by the functions $g_t \circ u_t$. As worked out in the certainty additive gauge, where the term $g_t \circ u_t = u_t^{\text{welf}}$ directly coincides with Bernoulli utility, the expression was given an interpretation of welfare (see chapter 7.3). As explained in chapter 7.4, fixing the unit of welfare makes the measure of absolute intertemporal risk aversion (or risk aversion on welfare) unique, while fixing the 'zero welfare level' makes the measure of relative intertemporal risk aversion unique. Precisely, the following assertions hold.

Lemma 5: Let there be given a sequence of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ satisfying axioms A1-A3, A4' and A5'. In addition, choose
i) a time $t^* \in \{1, ..., T\}$ and number $w^* \in \mathbb{R}_{++}$,
ii) outcomes x_t^{zero} for all $t \in \{1, ..., T\}$ or
iii) numbers $w_t \in \mathbb{R}$ for all $t \in \{1, ..., T\}$.
Then, for representations $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ in the sense of theorem 4 with twice differentiable functions $f_t \circ g_t^{-1}$ that
a) satisfy $\Delta G_{t^*} = w^*$, the risk measures AIRA_t
b) satisfy $g_t \circ u_t(x_t^{\text{zero}}) = 0$ for all $t \in \{1, ..., T\}$, the risk measures RIRA_t
c) satisfy $\Delta G_{t^*} = w^*$ and $g_t \circ u_t(x_t^{\text{zero}}) = 0$ for all $t \in \{1, ..., T\}$, the risk measures AIRA_t and RIRA_t
d) satisfy $\Delta G_{t^*} = w^*$ and $\underline{G}_t = w_t$ for all $t \in \{1, ..., T\}$, the risk measures AIRA_t and RIRA_t
are determined uniquely and independent of the choice of the Bernoulli utility functions for all $t \in \{1, ..., T\}$.

As all gauges of the representation correspond to particular choices of Bernoulli utility, lemma 5 can be restated as the fact that, once the corresponding welfare information has been fixed, the measures RIRA_t and AIRA_t are gauge invariant. In assertion a) the externally given information specifies the unit of welfare measurement, by prescribing a numerical value to the difference in welfare between the best and the worst outcome, i.e. $u_t^{\text{welf}}(x_t^{\max}) - u_t^{\text{welf}}(x_t^{\min}) = g_t \circ u_t(x_t^{\max}) - g_t \circ u_t(x_t^{\min}) = \overline{G}_t - \underline{G}_t = w^*$. Note that it is enough to fix the unit for some period, in order to determine the welfare unit for all periods.[11] Such a partial specification of the measure scale for welfare makes the measures of *absolute* intertemporal risk aversion unique. Assertion b) fixes the 'zero welfare level' for all periods, by prescribing outcomes in every period that shall correspond to a zero welfare. Note that such a specification of welfare leaves the

[11] The freedom in the choice of welfare unit for the different representations of the same preferences corresponds to the multiplicative constant a^g in the allowed transformations for g_t in the representation of theorem 4. As a^g is common to all periods, specifying a^g determines the unit of welfare measurement for all periods.

unit of welfare undetermined. The information is enough to render the measures of *relative* intertemporal risk aversion unique. Assertion c) fixes the welfare unit and the zero welfare level together. This step eliminates the freedom in the measure scale of welfare, going along with different representations, completely. In consequence, both measures of intertemporal risk aversion are determined uniquely. Assertion d) offers an alternative way to eliminate the indeterminacy of the measure scale for welfare. Here I fix the unit of welfare as in a), and prescribe numerical values to the worst outcomes in every period. Similarly, one could prescribe numerical values to any other arbitrary outcome in every period. The necessity to fix the 'zero welfare level', or specify some alternative information, in *every* period, is due to the allowed non-stationarity of preferences, which allows the functions $f_t \circ g_t^{-1}$ to vary arbitrarily over time. The next chapter gives conditions under which the functions g_t and f_t for different periods are directly related.

Chapter 9

Stationarity

9.1 Certainty Stationarity

The representation developed in the preceding chapter allows time and uncertainty aggregation to vary arbitrarily from period to period. This chapter introduces different stationarity assumptions for preferences and elaborates their implications for the representation. First, the current section develops an axiom restricting choice under certainty that renders intertemporal aggregation and Bernoulli utility stationary. On certain consumption paths, it gives rise to the common discount utility representation. Due to the finite planning horizon in my analysis, the corresponding axiom slightly differs from its usual formulation. It can be interpreted as a combination of the assumption that the mere passage of time does not change preference, and an assumption that the ranking of two lotteries does not depend on a common certain outcome in the last period. Subsequently, in section 9.2, I introduce an axiom that additionally makes uncertainty aggregation invariant over time. In particular, the resulting representation contains the model of generalized isoelastic preferences, usually employed to disentangle (atemporal) risk aversion from intertemporal substitutability. Section 9.3 works out an alternative stationarity assumption for the evaluation of uncertainty. Analogously to the axiom for certain consumption paths derived in the current section, it builds on the assumption that the ranking of two lotteries does not depend on a common certain outcome in the last period. An attractive feature of the resulting representation is that it condenses information on intertemporal risk aversion into a single parameter. The latter insight is worked out in section 9.4. There, I characterize intertemporal risk aversion in the different representation theorems given in this chapter. In addition, I state a simplified axiom characterizing intertemporal risk aversion of preferences in a stationary setting.

Stationarity, in the sense of the standard discount utility model is a ubiquitous assumption in economic modeling, and in particular in environmental economics. However, to the best of my knowledge, the assumption is expressed in terms of the underlying preference relations only for models featuring an infinite time horizon. In these models, the axiomatic characterization of stationarity requires that a decision maker prefers a consumption path x over another consumption path x' in the present, if and only if, he prefers a consumption path (x^0, x) over a consumption path (x^0, x') in the present (Koopmans 1960).[1] Such an axiomatization makes use of the fact that for an infinite time horizon, adding an additional outcome does not change the length of a consumption path. Precisely, both paths x and (x^0, x), are elements of X^∞ and can be compared by the same preference relation. On the contrary, for a finite time horizon, the paths x and (x^0, x) differ in length and, thus, cannot be compared by means of the same preference relation $\succeq$. The reason, why I keep my model in the finite time horizon is threefold. First, the common techniques for analyzing an infinite time horizon setting (contraction and fix point theorems) have to assume from the outset a positive rate of pure time preference. However, as mentioned in the introduction to this thesis, such a positive discount rate is not without controversy.[2] Second, the reasoning on stationarity carried out in this chapter together with the reasoning on attitude with respect to the timing of uncertainty resolution carried out in chapter 10, make a strong point for choosing a zero rate of time preference for a time consistent approach to choice under uncertainty. Third, for most planning processes and scenario evaluations there exists a reasonable upper bound for the planning horizon.[3] Considering the assumption of a finite horizon a competitive alternative to the assumption of an infinite planning horizon that is accompanied by a positive discount rate, I give a slightly different axiomatization of stationarity. In the discussion later on, I provide several comments with respect to the limit of an infinite time horizon. The following axiom is applicable in a finite time horizon setting and, there, yields the standard discount utility model for the evaluation of certain consumption paths.

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[1]See page 130 for details.

[2]Compare the citation of Ramsey (1928, 543), who states that a positive rate of pure time preference is "ethically indefensible". I come back to this issue in chapter 10.4.

[3]While such an upper bound can be in the magnitude of several decades, note that taking as upper bound a point of time by which our sun has burned out or turned into a red giant still provides a finite upper bound (Sackmann, Boothroyd & Kraemer 1993).

A7 (certainty stationarity) For all $\mathsf{x}^2, \mathsf{x}'^2 \in X^2$ and all $x^0 \in X$ holds

$$(\mathsf{x}^2, x^0) \succeq_1 (\mathsf{x}'^2, x^0) \quad \Leftrightarrow \quad \mathsf{x}^2 \succeq_2 \mathsf{x}'^2 .$$

On the right hand side of the equivalence, the decision maker faces a comparison between x^2 and x'^2 as consumption paths starting in period 2. Note that, by time consistency A5', the comparison on the right is equivalent to $(x^0, \mathsf{x}^2) \succeq_1 (x^0, \mathsf{x}'^2)$. On the left hand side of the equivalence, the decision maker faces a comparison between x^2 and x'^2 as consumption paths starting in period 1. The additional outcome x^0, which is commonly added to the paths x^2 and x'^2, makes (x^2, x^0) and (x'^2, x^0) choice objects of the appropriate length, so that they can be compared in period 1 by the preference relation $\succeq_1$. The important property of the axiom is that the decision maker's preference over the (certain) consumption paths is independent of their starting point.[4]

I want to give an interpretation of axiom 7 by separating the underlying idea into two steps. *Assume* that a decision maker in period 1, planning with time horizon T, prefers consumption plan (x^2, x^0) over plan (x'^2, x^0). Now, let him contemplate about his choice in period 2. Assume that he is confronted in period 2 with the exact same consumption paths (x^2, x^0) and (x'^2, x^0) (*not* with their continuation). Furthermore, let him anticipate that in period 2 he will plan ahead the same amount of periods as he does in period one, implying a time horizon $T+1$. Formally, I denote these preferences of the decision maker in period 2 with time horizon $T+1$ by $\succeq_{2|T+1}$. Then, given that nothing else changes between period 1 and period 2, I demand that the decision maker ranks (or rather plans to rank) the projects in both choice situations the same way. Requiring the latter for all consumption paths yields the condition

$$(\mathsf{x}^2, x^0) \succeq_{1|T} (\mathsf{x}'^2, x^0) \quad \Leftrightarrow \quad (\mathsf{x}^2, x^0) \succeq_{2|T+1} (\mathsf{x}'^2, x^0) \tag{9.1}$$

for all $\mathsf{x}^2, \mathsf{x}'^2 \in \mathsf{X}^2$ and $x^0 \in X$. Condition (9.1) most clearly captures the intuition of stationarity, in the sense that the mere passage of time should not change the evaluation. However, up to now the preference relations $\succeq_{\cdot|T}$ and $\succeq_{\cdot|T+1}$ are completely unrelated. In consequence, equation (9.1) on its own does not restrict the decision maker's preference relations $(\succeq_t)_{t\in\{1,\ldots,T\}} = (\succeq_{t|T})_{t\in\{1,\ldots,T\}}$ in any way. Thus, the second step in the reasoning has to relate the preference relation $\succeq_{2|T+1}$ to the relation $\succeq_2 = \succeq_{2|T}$. Both preference relations specify how the decision maker anticipates to evaluate choice objects in period 2. The relation $\succeq_{2|T}$ specifies his ranking when planning $T-2$ periods ahead (until period T), and the relation $\succeq_{2|T+1}$ states his ranking when he plans $T-1$ periods

[4]Note the difference to time consistency. The latter is a condition on consumption paths starting in the same period that yield a common outcome in the first period. Then, the *continuation* of the paths in the next period should be ranked the same way as the complete paths in the earlier period.

ahead (until period $T+1$). Accepting stationarity in the sense of equation (9.1), axiom A7 requires the following relation between $\succeq_{\cdot|T}$ and $\succeq_{\cdot|T+1}$:

$$\mathsf{x}^2 \succeq_{2|T} \mathsf{x}'^2 \quad \Leftrightarrow \quad (\mathsf{x}^2, x^0) \succeq_{2|T+1} (\mathsf{x}'^2, x^0) \tag{9.2}$$

for all $\mathsf{x}^2, \mathsf{x}'^2 \in \mathsf{X}^2$ and $x^0 \in X$. In words, if two scenarios or projects are evaluated with a time horizon of $T+1$, and yield the same outcome in period $T+1$, then, an evaluation based only on a time horizon T shall yield the same ranking of the scenarios.

Let me point out the analogous reasoning to yield stationarity from the assumption expressed in equation (9.1) for the case of an infinite planning horizon. Denote the consumption paths corresponding to (x^2, x^0) and (x'^2, x^0) by $\mathsf{x}, \mathsf{x}' \in X^\infty$. Then, by time consistency the right hand side of equation (9.1) is equivalent to $(x^0, \mathsf{x}) \succeq_{1|T+1} (x^0, \mathsf{x}')$ for all $\mathsf{x}, \mathsf{x}' \in X^\infty$ and $x^0 \in X$. Moreover, in the infinite horizon setting, it holds $\succeq_{1|T+1} = \succeq_{1|\infty} = \succeq_{1|T}$, a relation which makes equation (9.2) dispensable. That way, I arrive at the standard axiom of stationarity for the infinite planning horizon: $\mathsf{x} \succeq_{1|\infty} \mathsf{x}' \Leftrightarrow (x, \mathsf{x}) \succeq_{1|\infty} (x, \mathsf{x}')$ for all $x \in X$ and all $\mathsf{x}^\infty, \mathsf{x}'^\infty \in X^\infty$, dating back to Koopmans (1960)[294][5]. Hence, at first sight, the second assumption, corresponding to equation (9.1), seems to come for free with an infinite time horizon. However, this is not the case. It is a necessary assumption in the standard framework with an infinite planning horizon that the decision maker applies a positive rate of pure time preference. Therefore, the weight given to future consumption converges to zero. Thus, the assumption that coinciding outcomes in the 'last' period of the planning horizon do not matter for the ranking of consumption paths is implicit in the infinite horizon setting. It is the combined result of the decision maker's intrinsic devaluation of the future and his infinite planning horizon.

As in the simplified setting analyzed in chapter 6.3, also in the general stationary setting the sets of Bernoulli utility functions coincide for different periods. Therefore, define $u \in B_{\succeq} \equiv B_{\succeq_1}$. Preference stationarity on certain consumption paths as formulated in axiom A7, together with the assumptions of the previous chapter, yields the following representation.

Theorem 7: Let there be given a sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ and a Bernoulli utility function $u \in B_{\succeq}$ with range U. The sequence $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting)

[5]Koopmans (1960) actually formulates his postulates in terms of utility functionals. However the translation of his postulate 4 into the preference setup is immediate. His general axiomatic setting is translated into preferences in Koopmans (1972), again with stationarity corresponding to postulate 4.

ii) A4' for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) A7 (certainty stationarity)

if and only if, there exist strictly increasing and continuous functions $f_t : U \to \mathbb{R}$ for all $t \in \{1, ..., T\}$ and $g : U \to \mathbb{R}$ as well as a discount factor $\beta \in \mathbb{R}_{++}$, such that with defining

v) the normalized discount weights

$$\beta_t = \beta \, \frac{1-\beta^{T-t}}{1-\beta^{T-t+1}} \text{ for } \beta \neq 1 \text{ and}$$

$$\beta_t = \frac{T-t}{T-t+1} \qquad \text{for } \beta = 1 \text{ and}$$

vi) the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by $\tilde{u}_T(x_T) = u(x_T)$ and recursively

$$\tilde{u}_{t-1} = g^{-1} \left[(1 - \beta_{t-1}) \, g \circ u(x_{t-1}) + \beta_{t-1} \, g \circ \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \right] \tag{9.3}$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t) \quad \forall \, p_t, p'_t \in P_t.$$

Moreover the functions g and f_t are unique up to nondegenerate positive affine transformations.

Certainty stationarity implies that the same Bernoulli utility function $u \in B_{\succeq}$ can be used in the representation for all periods. Moreover, it makes the functions g_t, characterizing intertemporal aggregation, coincide for adjacent periods up to a (common) multiplicative constant. This constant corresponds to the discount factor β. As shown in the proof, a representation $(u, f_t, g)_{t \in \{1,...,T\}}$ for $(\succeq_t)_{t \in \{1,...,T\}}$ in the sense of theorem 7 corresponds to a representation $(u_t, f_t, g_t)_{t \in \{1,...,T\}} = (u, f_t, \beta^t g)_{t \in \{1,...,T\}}$ for $(\succeq_t)_{t \in \{1,...,T\}}$ in the sense of the general non-stationary representation of theorem 4. Expressing this relation in words, the information characterizing intertemporal aggregation, which in the general setting is contained in the functions g_t for $t \in \{1, ..., T\}$, can be captured in the stationary setting by two quantities. The first piece of information is taken up by the function g, which now is common to all periods. In the one commodity Epstein Zin gauge, it comprises the information on intertemporal substitutability. The second piece of information characterizes the change of the functions g_t between different periods. This change is captured in a single parameter, the discount factor β, which describes the reduction in weight given to future outcomes.

For the limit of an infinite time horizon under the assumption $\beta < 1$, the normalized discount weights β_t used in the representation converge to the discount factor itself: $\lim_{T\to\infty} \beta_t = \beta$ for all t. Then, the weight given to the present as opposed to the future is constant. However, for a decision maker who plans with a finite time horizon, the weights β_t have to accommodate not only discounting, but also the weight that an individual period receives as opposed to the remaining future. The shorter the time horizon, or the closer the end of the time horizon, the higher must be the weight that the present period obtains as opposed to the remaining future.[6] On certain consumption paths the evaluation is ordinally equivalent to the representation

$$\mathsf{x}^t \succeq_t \mathsf{x}'^t \quad \Leftrightarrow \quad \sum_{\tau=t}^{T} \beta^t g \circ u(\mathsf{x}^t_\tau) \geq \sum_{\tau=t}^{T} \beta^t g \circ u(\mathsf{x}'^t_\tau).$$

Moreover, in the $g = \text{id}$ gauge, the recursive construction of aggregate utility in equation (9.3) becomes

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = (1 - \beta_{t-1})\, u(x_{t-1}) + \beta_{t-1}\, \mathcal{M}^{f_t}(p_t, \tilde{u}_t)$$

and yields for certain consumption paths a representation ordinally equivalent to the widely applied evaluation

$$\mathsf{x}^t \succeq_t \mathsf{x}'^t \quad \Leftrightarrow \quad \sum_{\tau=t}^{T} \beta^t u(\mathsf{x}^t_\tau) \geq \sum_{\tau=t}^{T} \beta^t u(\mathsf{x}'^t_\tau).$$

In difference to the intertemporal aggregation rules, the uncertainty aggregation rules are allowed to vary arbitrarily over time. The next two sections elaborate two different assumptions rendering the uncertainty aggregation rules stationary as well.

9.2 Risk Stationarity I

In the preference framework of the preceding section, I assume stationarity in the evaluation of certain consumption paths. The assumption implies a close relation between intertemporal aggregation rules in different periods. In contrast, in the representation of theorem 7 uncertainty evaluation is allowed to vary arbitrarily over time.[7] It stands

[6] In particular, at the end of the time horizon, the weight given to the future has to be zero. Note, that this reasoning is necessary because the weights given to the present and to the future have to add up to one in the time aggregator $g^{-1}\left[(1 - \beta_{t-1})\, g(\cdot) + \beta_{t-1}(\cdot)\right]$. Otherwise the symmetric characterization of intertemporal aggregation and uncertainty aggregation by functions g_t and f_t would fail.

[7] Precisely, uncertainty evaluation is allowed to vary arbitrarily between different periods t and t'. By the requirement of time consistency, uncertainty aggregation has to be fixed for a given period t and, thus, independent of whether period t is τ or τ' periods into the future.

to reason that a decision maker who relates his evaluation of certain consumption paths between different periods, is also willing to relate his evaluation of uncertain consumption plans for different periods. An example of a preference representation which relates uncertainty evaluation between different periods is the generalized isoelastic model. It has been discussed in chapter 7.1 as the common framework to disentangle (atemporal) risk aversion from intertemporal substitutability. In the multiperiod framework, the generalized isoelastic model features uncertainty aggregation rules that are commonly characterized by $f_t = z^\alpha \, , \forall \, t \in \{1, ..., T\}$.

In order to state an axiom that implies time constant uncertainty aggregation rules, it proves useful to introduce a special notation for constant consumption paths. Let $\bar{\mathrm{x}}^t = (\bar{x}, \bar{x}, ..., \bar{x})$ denote the certain constant consumption path that gives consumption $\bar{x}$ from t until T. Then $\frac{1}{2}\bar{\mathrm{x}}^t + \frac{1}{2}\bar{\mathrm{x}}'^t \in P_t$ is the lottery in period t that randomizes with probability $\frac{1}{2}$ between the constant consumption streams giving $\bar{x}$ and $\bar{x}'$. The following axiom demands that these randomized consumption streams relate to certain consumption streams the same way in different periods.

A8 (risk stationarity I) For all $t \in \{1, ..., T-1\}$ holds

$$\tfrac{1}{2}\bar{\mathrm{x}}^{\,t} + \tfrac{1}{2}\bar{\mathrm{x}}'^{\,t} \succeq_t \bar{\mathrm{x}}''^{\,t} \quad \Leftrightarrow \quad \tfrac{1}{2}\bar{\mathrm{x}}^{\,t+1} + \tfrac{1}{2}\bar{\mathrm{x}}'^{\,t+1} \succeq_{t+1} \bar{\mathrm{x}}''^{\,t+1} \quad \forall \, \bar{x}, \bar{x}', \bar{x}'' \in X \, .$$

The axiom can be conceived as an indifference requirement to the start, and thus, the duration of a taken risk. In particular, for a decision maker who is indifferent between the lottery $\frac{1}{2}\bar{x} + \frac{1}{2}\bar{x}'$ and a certain outcome $\bar{x}''$ in period T, axiom A8 requires that he is indifferent between the lottery $\frac{1}{2}(\bar{x}, \bar{x}) + \frac{1}{2}(\bar{x}', \bar{x}')$ and the certain consumption path $(\bar{x}'', \bar{x}'')$ in period $T-1$ as well. Be aware that in the lotteries of axiom A8 the outcomes in the different periods are perfectly correlated. In particular in the above example, lottery $\frac{1}{2}(\bar{x}, \bar{x}) + \frac{1}{2}(\bar{x}', \bar{x}')$ is not the same as the lottery $\frac{1}{2}(\bar{x}, \frac{1}{2}\bar{x} + \frac{1}{2}\bar{x}') + \frac{1}{2}(\bar{x}', \frac{1}{2}\bar{x} + \frac{1}{2}\bar{x}')$, which would correspond to independent coin tosses in both periods. Adding axiom A8 to the assumptions of theorem 7 yields the following representation.

Theorem 8: Let there be given a sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ and a Bernoulli utility function $u \in B_\succeq$ with range U. The sequence $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting)

ii) A4' for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) A7-A8 (stationarity)

if and only if, there exist strictly increasing[8] and continuous functions $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ as well as a discount factor $\beta \in \mathbb{R}_{++}$, such that with defining

v) the normalized discount weights

$$\beta_t = \beta \, \frac{1-\beta^{T-t}}{1-\beta^{T-t+1}} \text{ for } \beta \neq 1 \text{ and}$$

$$\beta_t = \frac{T-t}{T-t+1} \qquad \text{for } \beta = 1 \text{ and}$$

vi) the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by $\tilde{u}_T(x_T) = u(x_T)$ and recursively

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = g^{-1} \left[(1 - \beta_{t-1})\, g \circ u(x_{t-1}) + \beta_{t-1}\, g \circ M^f(p_t, \tilde{u}_t) \right] \tag{9.4}$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^f(p_t, \tilde{u}_t) \geq \mathcal{M}^f(p'_t, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t.$$

Moreover g and f are unique up to nondegenerate positive affine transformations.

In this representation a common function f characterizes risk attitude in all periods. Relating the representation to the general non-stationary setting, a representation (u, f, g) in the sense of theorem 8 corresponds to the representation $(u, f, \beta^t g)$ in the sense of theorem 4. Due to the fact that both functions, g and f, apply to all periods, gauging can be carried out as already discussed in chapter 6.4.

Lemma 6: Gauge lemma 1, corollary 2 (f-gauge) and corollary 3 (g-gauge) of section 6.4 also hold for the multiperiod representation of theorem 8.
Precisely, replace in the respective statements 'theorem 2' by 'theorem 8' and '$i - iii$)' by '$i - iv$)'. Moreover, either replace the word 'monotonic' by 'increasing' in corollaries 2 and 3, or replace the word 'increasing' by 'monotonic' in theorem 8.

In the one-commodity Epstein Zin gauge ($u = \text{id}$), the representation yields a multiperiod extension of the generalized isoelastic model discussed in chapter 7.1. For $X \subset \mathbb{R}$, the Epstein Zin gauge for the isoelastic setting, where $f = z^\alpha$ and $g(z) = z^\rho$, brings about the following recursive construction of aggregate utility $\tilde{u}_t$.

Equation (9.4) in the *Epstein Zin gauge* ($u = \text{id}$) with isoelastic preferences:

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = \left\{ (1 - \beta_{t-1})\, x^\rho + \beta_{t-1} \left[\int_{\tilde{X}_t} \tilde{u}_t(\tilde{x}_t)^\alpha \, dp_t \right]^{\frac{\rho}{\alpha}} \right\}^{\frac{1}{\rho}} \tag{9.5}$$

[8] Alternatively, the theorem can be stated with 'increasing' replaced by 'monotonic' (see chapter 6.3).

Analogous formulations have been used in many applications in order to disentangle atemporal risk aversion, characterized by α, from intertemporal substitutability, characterized by ρ. For details and an overview over the respective literature consult chapter 7.1. Recall that I have introduced the functions u_t and $\tilde{u}_t$ as explicit arguments in the uncertainty aggregation rules, in order to analyze and make use of the freedom in the choice of Bernoulli utility. In the literature disentangling risk aversion from intertemporal substitutability, the uncertainty evaluation is usually stated in terms of the power mean, which corresponds to $\mathcal{M}^\alpha$, but uses the probability measure p_t^* induced by p_t on U_t through the function $\tilde{u}_t$ (see chapter 5.3, in particular footnote 29). Then, denoting the elements of $U_t \subset \mathrm{I\!R}$ by $\mathfrak{u}_t$, equation (9.5) writes as

$$\tilde{u}_{t-1}(x_{t-1}, p_t^*) = \left\{ (1-\beta_{t-1})\, x^\rho + \beta_{t-1} \left[\int_{U_t} \mathfrak{u}_t{}^\alpha \, dp_t^* \right]^{\frac{\rho}{\alpha}} \right\}^{\frac{1}{\rho}} .$$

The standard form of the aggregator is obtained for the limit of an infinite time horizon, where $\lim_{T\to\infty} \beta_t = \beta$ for all t.

9.3 Risk Stationarity II

Stationarity of risk attitude in the preceding section was primarily motivated by the objective to obtain constant uncertainty aggregation rules. In this section, I reconsider stationarity of preference in a risky world. Departing from axiom A7 for certain consumption paths, I derive an alternative requirement for stationarity of risk attitude, yielding a preference representation distinct from that given in theorem 8.

In section 9.1 I have motivated the axiom of certainty stationarity by splitting it up into two assumptions. The first requirement, corresponding to equation (9.1), expresses that the mere passage of time shall not change preferences. The second assumption, corresponding to equation (9.2), compares two scenarios yielding the same outcome in period $T+1$. For such scenarios, it requires that adapting a time horizon of $T+1$ or of T shall yield the same ranking of the two scenarios. In the following, I give an analogous reasoning for risky scenarios. As already in axioms A6 and A8, it proves sufficient to require risk stationarity only for 'coin toss' compositions of certain consumption paths, i.e. probability a half mixtures of type $\frac{1}{2}\mathrm{x} + \frac{1}{2}\mathrm{x}'$. Moreover, it is enough to have the decision maker rank these lotteries with respect to certain alternatives. Then, the analogue requirement to equation (9.1) becomes

$$\tfrac{1}{2}(\mathrm{x}, x^0) + \tfrac{1}{2}(\mathrm{x}', x^0) \succeq_{t|T} (\mathrm{x}'', x^0) \;\Leftrightarrow\; \tfrac{1}{2}(\mathrm{x}, x^0) + \tfrac{1}{2}(\mathrm{x}', x^0) \succeq_{t+1|T+1} (\mathrm{x}'', x^0) \qquad (9.6)$$

for all $\mathrm{x}, \mathrm{x}', \mathrm{x}'' \in \mathrm{X}^{t+1}$, $x^0 \in X$ and $t \in \{1, ..., T\}$. In words, the mere passage of time shall not change the ranking between the different scenarios. As I want the decision maker to evaluate lotteries where uncertainty resolves at any point in the future, I require equation (9.6) to hold for all periods.[9]

The second step to arrive at the axiom of risk stationarity, is to relate the relations $\succeq_{\cdot|T}$ and $\succeq_{\cdot|T+1}$. As in section 9.1, I require that scenarios whose outcomes coincide in the last period of a finite planning horizon $T+1$ shall be ranked the same way when applying a planning horizon of T. This demand is formalized by the statement

$$\tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}' \succeq_{t+1|T} \mathsf{x}'' \quad \Leftrightarrow \quad \tfrac{1}{2}(\mathsf{x}, x^0) + \tfrac{1}{2}(\mathsf{x}', x^0) \succeq_{t+1|T+1} (\mathsf{x}'', x^0) \tag{9.7}$$

for all $\mathsf{x}, \mathsf{x}', \mathsf{x}'' \in \mathsf{X}^{t+1}$, $x^0 \in X$ and $t \in \{1, ..., T\}$. As the right hand side of the requirements in equations (9.6) and (9.7) coincides, together, the equations bring about the following axiom for stationarity of risk attitude in a setting with a finite planning horizon.

A9 (risk stationarity II) For all $t \in \{1, ..., T-1\}$ and $x^0 \in X$:
$$\tfrac{1}{2}(\mathsf{x}, x^0) + \tfrac{1}{2}(\mathsf{x}', x^0) \succeq_t (\mathsf{x}'', x^0) \quad \Leftrightarrow \quad \tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}' \succeq_{t+1} \mathsf{x}'' \quad \forall\, \mathsf{x}, \mathsf{x}', \mathsf{x}'' \in \mathsf{X}^{t+1}.$$

In short, the decision maker ranks lotteries of the form $\frac{1}{2}\mathsf{x} + \frac{1}{2}\mathsf{x}'$ the same way when they are faced in period t as when they are faced in period $t+1$. When facing them in period t, the additional outcome x^0 at the end of the planning horizon, which coincides for all consumption paths, does not change his ranking.

Before I come to the representation, let me briefly point out the analogous reasoning to yield risk stationarity from the assumption expressed in equation (9.6) in the case of an infinite planning horizon. Denote the consumption paths corresponding to (x, x^0) and (x', x^0) simply by $\mathsf{x}, \mathsf{x}' \in X^\infty$, yielding the notation $\frac{1}{2}\mathsf{x}+\frac{1}{2}\mathsf{x}'$ for the lotteries considered in the infinite horizon version of equation (9.6). Moreover, in the infinite horizon setting, it is $\succeq_{1|T+1} = \succeq_{1|\infty} = \succeq_{1|T}$. Then, by time consistency, equation (9.6) for $t = 1$ is equivalent to

$$\tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}' \succeq_{1|\infty} \mathsf{x}'' \quad \Leftrightarrow \quad (x_1, \tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}') \succeq_{1|\infty} (x_1, \mathsf{x}'')$$

for all $\mathsf{x}, \mathsf{x}', \mathsf{x}'' \in X^\infty$ and $x_1 \in X$. Similarly for $t = 2$ equation (9.6) is equivalent to

$$(x_1, \tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}') \succeq_{1|\infty} (x_1, \mathsf{x}'') \quad \Leftrightarrow \quad (x_1, x_2, \tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}') \succeq_{1|\infty} (x_1, x_2, \mathsf{x}'')$$

for all $\mathsf{x}, \mathsf{x}', \mathsf{x}'' \in X^\infty$ and $x_1, x_2 \in X$. The latter statement for $t = 2$ can be transformed

[9] Alternatively, I could formulate the requirement analogously to equation (9.1) in section 9.1 by only considering preference in periods 1 and 2. Such a reformulation is straight forward, once it is recognized that time consistency A4' allows to carry over all the requirements in equation (9.6) into the first two periods, by adding common outcomes to the beginning of all consumption plans which start in later periods.

using the corresponding statement for $t = 1$ into the requirement:

$$\tfrac{1}{2}\mathrm{x} + \tfrac{1}{2}\mathrm{x}' \succeq_{1|\infty} \mathrm{x}'' \quad \Leftrightarrow \quad (x_1, x_2, \tfrac{1}{2}\mathrm{x} + \tfrac{1}{2}\mathrm{x}') \succeq_{1|\infty} (x_1, x_2, \mathrm{x}'')$$

for all $\mathrm{x}, \mathrm{x}', \mathrm{x}'' \in X^\infty$ and $x_1, x_2 \in X$. By induction I obtain the general requirement

$$\tfrac{1}{2}\mathrm{x} + \tfrac{1}{2}\mathrm{x}' \succeq_{1|\infty} \mathrm{x}'' \quad \Leftrightarrow \quad (\mathrm{x}^t, \tfrac{1}{2}\mathrm{x} + \tfrac{1}{2}\mathrm{x}') \succeq_{1|\infty} (\mathrm{x}^t, \mathrm{x}'') \tag{9.8}$$

for all $\mathrm{x}, \mathrm{x}', \mathrm{x}'' \in X^\infty$, $t \in \mathbb{N}$ and $\mathrm{x}^t \in X^t$. A corresponding[10] axiom for stationarity of risk attitude is found in Chew & Epstein (1991, 356).

Preference stationarity for the evaluation of lotteries as formulated in axiom A9, together with the assumptions of the previous chapter, yields the following representation.

Theorem 9: Let there be given a sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ and a Bernoulli utility function $u \in B_\succeq$ with range U. The sequence $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting)

ii) A4' for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) A9 (risk stationarity II)

if and only if, there exists a strictly increasing and continuous function $g : U \to \mathbb{R}$ and a discount factor $\beta \in \mathbb{R}_{++}$ as well as a function $h \in \left\{\exp, \mathrm{id}, \frac{1}{\exp}\right\}$, such that with defining the functions $\tilde{w}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by

v) $\tilde{w}_T(x_T) = g \circ u(x_T)$ and recursively

$$\tilde{w}_{t-1}(x_{t-1}, p_t) = g \circ u(x_{t-1}) + \beta\, \mathcal{M}^h(p_t, \tilde{w}_t) \ \text{ or by} \tag{9.9}$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p_t' \quad \Leftrightarrow \quad \mathcal{M}^h(p_t, \tilde{w}_t) \geq \mathcal{M}^h(p_t', \tilde{w}_t) \quad \forall\, p_t, p_t' \in P_t. \tag{9.10}$$

Moreover, if the representation employs $h \in \left\{\exp, \frac{1}{\exp}\right\}$, then two functions g and g' both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, there exists $b \in \mathbb{R}$ such that $g = g' + b$. In a representation employing $h = \mathrm{id}$, two functions g and g' both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, there exist $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $g = ag' + b$.

[10] In difference to the above formulation, the authors require condition (9.8) for all lotteries, not just for the probability a half ('coin toss') combinations that I have used and which prove sufficient in my setting.

The representation constructed in theorem 9 slightly differs from earlier representations. First, the functions employed for the recursive construction of the representation in equations (9.9) and (9.10) are not complete analogues to those used in earlier representations. Precisely, the functions $\tilde{w}_t$ used in equation (9.9) relate to the functions $\tilde{u}_t$ used in the earlier theorems as $\tilde{w}_t = \frac{1}{1-\beta_t}\, g \circ \tilde{u}_t$. Second, instead of the functions f_t, a function h characterizes the uncertainty aggregation rule employed above. The function h closely connects to *intertemporal* risk aversion and, in particular, characterizes whether intertemporal risk aversion, risk neutrality or risk seeking prevail. Note that for $h = \frac{1}{\exp}$ the characterization of the uncertainty aggregation rule corresponds to the function $h(z) = \frac{1}{\exp(z)} = \exp(-z)$.[11]

This departure from earlier layouts of the representations is caused by axiom A9. The latter implies a close relation between the function g characterizing intertemporal aggregation, and the functions f_t characterizing uncertainty aggregation. In order to exploit this relation, the functions $\tilde{w}_t$ and h are introduced (confer part four of the proof to see how this simplifies the representation). In difference to the requirements of intertemporally additive expected utility, the relation implied by axiom A9 between 'risk aversion' and 'intertemporal substitutability'[12] leaves one degree of freedom.[13] This freedom breaks the representation up into the three classes, corresponding to $h \in \left\{\exp, \text{id}, \frac{1}{\exp}\right\}$. In each of these classes, the functions f_t, characterizing uncertainty aggregation in the sense of the earlier representation theorems, are affine transformations of $h \circ g$.[14] However, the mentioned relation between f_t and g, which is used to simplify the representation, only holds for particular choices of $g \in \hat{g}$. In consequence, in order to exploit the relation, I have to give up part of the affine freedom for the choice of g. For this reason, in the cases where $h \in \left\{\exp, \frac{1}{\exp}\right\}$, the function g is no longer free up to affine transformations, but only up to a translational constant.

Another consequence of incorporating the function g, characterizing intertemporal aggregation, into the function $\tilde{w}_t$, is that equation (9.9) is time additive in $g \circ u$. Note that $g \circ u$ corresponds, up to discounting, to the time additive Bernoulli utility function defined as welfare in chapter 7.3. Thus, for the certainty additive gauge ($g = \text{id}$) equation

[11] I avoid the notation $h = \exp^{-1}$ because h^{-1} is used to denote the inverse.

[12] I put quotation marks, as it has been analyzed in chapter 7.1 that this interpretation of f and g is meaningful only in a one commodity setting applying the Epstein Zin gauge.

[13] Recall that in the intertemporally additive expected utility model, the coefficient of relative risk aversion is always fixed to the inverse of the elasticity of intertemporal substitution.

[14] This is why h in the above representation is closely related to the functions $f_t \circ g_t^{-1}$ characterizing intertemporal risk aversion in the earlier representations. However, note that the affine transformation is negative for the case where $h = \frac{1}{\exp}$. The next section works out the precise relation.

(9.9) is linear in Bernoulli utility u^{welf}. In that case the recursive representation employing $\tilde{w}_t$ and the one using $\tilde{u}_t$ coincide up to the factor $1-\beta_t$.[15] In theorem 9, there seem to be three disconnected classes of representations corresponding to $h \in \left\{\exp, \text{id}, \frac{1}{\exp}\right\}$. The exploration of intertemporal risk aversion carried out in the next section naturally gives rise to a continuous connection between these different representations.

9.4 Intertemporal Risk Aversion

The section analyzes how intertemporal risk aversion is characterized in the representations developed in this section. As the stationary setting is a special case of the general multiperiod setup developed in chapter 8, the axiomatic definition of intertemporal risk aversion given in chapter 8.2 applies. However, alternatively, a slightly less demanding axiom can be used. In a stationary setting, it proves sufficient to compare lotteries to constant consumption paths only. Such a definition states that a decision maker exhibits *weak intertemporal risk aversion* in period t, if and only if, the following axiom is satisfied:

A6$^{\text{w}}_{\text{st}}$ (weak intertemporal risk aversion) For all $\bar{\mathrm{x}}^t, \mathrm{x}^t \in \mathrm{X}^t$ holds

$$\bar{\mathrm{x}}^t \sim_t \mathrm{x}^t \quad \Rightarrow \quad \bar{\mathrm{x}}^t \succeq_t \sum_{i=t}^{T} \frac{1}{T-t+1} \left(\bar{\mathrm{x}}^t_{-i}, \mathrm{x}^t_i\right).$$

A decision maker is said to exhibit *strict intertemporal risk aversion* in period t, if and only if, the following axiom is satisfied:

A6$^{\text{s}}_{\text{st}}$ (strict intertemporal risk aversion) For all $\bar{\mathrm{x}}^t, \mathrm{x}^t \in \mathrm{X}^t$ it holds

$$\bar{\mathrm{x}}^t \sim_t \mathrm{x}^t \quad \wedge \quad \exists\, \tau \in \{t, ..., T\} \text{ s.th. } [\mathrm{x}^t_\tau]_\tau \not\sim_\tau [\bar{x}]_\tau$$

$$\Rightarrow \quad \bar{\mathrm{x}}^t \succ_t \sum_{i=t}^{T} \frac{1}{T-t+1} \left(\bar{\mathrm{x}}^t_{-i}, \mathrm{x}^t_i\right).$$

The interpretation of the axioms is analogous to that of axioms A6$^{\text{w}}$ and A6$^{\text{s}}$ discussed at length in chapter 7.2. The only difference is that, as $\bar{\mathrm{x}}^t$ is a constant consumption path, the second part of the premise in the strict version simply requires the consumption path x to exhibit some variation. Then, the second line of the axiom demands that the constant consumption path delivering $\bar{x}$ in every period, is preferred to a lottery whose outcome paths differ in one period from $\bar{x}$. In that respective period i, which is drawn with equal probability from $\{t, ..., T\}$, the outcome $\bar{x}$ is replaced by outcome x_i. For

[15] The general $g-$gauge is obtained simply by eliminating the function g from equation (9.9) and abandoning the freedom to pick the Bernoulli utility function u freely. The next section elaborates more gauges.

some i, the outcome x_i is preferred, and for others it is judged inferior with respect to the outcome $\bar{x}$ prevailing in the other periods. The axiom requires the decision maker to prefer the constant consumption path $\bar{\mathsf{x}}^t$ over the described lottery, which might yield a better outcome than $\bar{x}$ in some period, but might as well yield a worse outcome. Again, the outcomes that are judged better than $\bar{x}$ and those that are judged inferior with respect to $\bar{x}$ balance in the sense that receiving all (i.e. consumption path x), would make the decision maker indifferent to the constant path. However, the lottery only gives the decision maker one of the outcomes taken from x. Thus, the fear of receiving an outcome which is judged inferior with respect to $\bar{x}$ makes the intertemporally risk averse decision maker prefer the certain and constant consumption path to the lottery.

If axiom A6$^{\text{s}}_{\text{st}}$ (A6$^{\text{w}}_{\text{st}}$) is satisfied with $\succ_t$ ($\succeq_t$) replaced by $\prec_t$ ($\preceq_t$), the decision maker is called a strong (weak) *intertemporal risk seeker*. If a decision maker's preferences satisfy weak intertemporal risk aversion as well as weak intertemporal risk seeking, the decision maker is called *intertemporally risk neutral*. The following theorem relates intertemporal risk attitude to the functional representation of theorem 7. It is similar to theorem 6 worked out in chapter 8, however, it uses the weaker axioms A6$^{\text{w}}_{\text{st}}$ and A6$^{\text{s}}_{\text{st}}$ adapted to the stationary setting.

Theorem 10: Let the sequence of triples $(u, f_t, g)_{t\in\{1,...,T\}}$ represent the set of preferences $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 7. Furthermore let $t \in \{1, ..., T-1\}$. Then the following assertions hold:

a) A decision maker is strictly intertemporally risk averse [seeking] in period t in the sense of axiom A6$^{\text{s}}_{\text{st}}$, if and only if, $f_t \circ g^{-1}(z)$ is strictly concave [convex] in $z \in \Gamma_t$.

b) A decision maker is weakly intertemporally risk averse [seeking] in period t in the sense of axiom A6$^{\text{w}}_{\text{st}}$, if and only if, $f_t \circ g^{-1}(z)$ is concave [convex] in $z \in \Gamma_t$.

c) A decision maker is intertemporally risk neutral in period t, if and only if, $f_t \circ g^{-1}(z)$ is linear in $z \in \Gamma_t$.

d) The above assertions hold as well, if axiom A6$^{\text{s}}_{\text{st}}$ is replaced by axioms A6$^{\text{s}}$ or A6$^{\text{s}}_{*}$, and if axiom A6$^{\text{w}}_{\text{st}}$ is replaced by axioms A6$^{\text{w}}$ or A6$^{\text{w}}_{*}$.

Intertemporal risk attitude is described by the second order characteristics of the function $f_t \circ g^{-1}(z)$. I refer to the latter as the stationary characterization of intertemporal risk attitude. It excludes the discount rate β, which enters the general expression characterizing intertemporal risk attitude $f_t \circ g_t^{-1}(z) = f_t \circ g^{-1}(\beta^{-t} z)$. Compare proposition 9 in section 7.2 to find that $f_t \circ g^{-1}$ is convex, if and only if $f_t \circ g_t^{-1}$ is convex.[16] In the

[16]Of course this statement also follows immediately from a comparison of theorem 10 with its non-

certainty stationary setting of representation theorem 7, the functions f_t are allowed to vary arbitrarily over time. Therefore, the decision maker's intertemporal risk attitude may also differ arbitrarily between different periods. This feature changes for the representations worked out in sections 9.2 and 9.3, which assume stationarity of preference also for risky choices. Concerning the representation of theorem 8, observe that it is the special case of theorem 7, where uncertainty aggregation is constant over time, i.e. $f_t = f \, \forall \, t \in \{1, ..., T\}$. In consequence, theorem 10 applies with $f_t g^{-1} = f g^{-1}$ independent of the period. Thus, the decision maker is either intertemporally risk averse, risk neutral or risk seeking in *all* periods. The same is true for a decision maker deciding in accordance with risk stationarity II as formulated in axiom A9. I have pointed out that h in the representation of theorem 9 corresponds, for an adequate choice of $g \in \hat{g}$, to an affine transformation of $f_t \circ g^{-1}$. Therefore, the decision maker is intertemporally risk averse, risk neutral or risk seeking depending on whether h is respectively $\frac{1}{\exp}$, id or exp.[17]

For a quantitative characterization of risk attitude, the definitions of absolute and relative intertemporal risk aversion given in chapter 8.2 apply. In order to render these risk measures unique, I first have to specify the range or, alternatively, the unit and the zero level of welfare $u^{\text{welf}} = g \circ u$. Other than for the non-stationary setting worked out in chapter 8, it will suffice to fix the measure scale of welfare for one period in order to determine it, and thus the coefficients of intertemporal risk aversion, for all periods. However, in difference to the stationary analysis without discounting in chapter 7.4, I have to decide for which period I fix the measure scale, e.g. the range of u_t^{welf} to some given interval G^*. I adopt the convention to fix the measure scale of welfare in the stationary setting always for the first period. In consequence, fixing $\text{range}(u_1^{\text{welf}}) = G^*$ implies that the range of welfare for later periods is fixed to $\text{range}(u_t^{\text{welf}}) = \beta^{t-1}\text{range}(u_1^{\text{welf}}) = \beta^{t-1}G^*$. The following adaption of lemma 5 to the stationary setting applies.

Lemma 7: Let there be given a sequence of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ satisfying axioms A1-A3, A4', A5' and A7 or A9. In addition, choose
i) a number $w^* \in \mathbb{R}_{++}$,
ii) an outcome $x^{\text{zero}} \in X$ or
iii) a nondegenerate closed interval $W^* \subset \mathbb{R}$.
Then, for representations in the sense of theorems 7 or 8 with twice differentiable

stationary analogue, theorem 6.

[17] Note again that in the case where $h = \frac{1}{\exp}$, h is a negative affine transformation of $f_t \circ g^{-1}$, making the latter concave (compare footnote 14). Also note that intertemporal risk attitude going along with preferences that satisfy axiom A9 can be observed better in corollary 8 following below.

functions $f_t \circ g^{-1}$ and for representations in the sense of theorem 9, which
a) satisfy $\Delta G = w^*$, the risk measures AIRA_t
b) satisfy $g \circ u(x^{\text{zero}}) = 0$, the risk measures RIRA_t
c) satisfy $\Delta G = w^*$ and $g \circ u(x^{\text{zero}}) = 0$, the risk measures AIRA_t and RIRA_t
d) satisfy $G = W^*$, the risk measures AIRA_t and RIRA_t
are determined uniquely and independent of the choice of the Bernoulli utility function.

Again, independence of Bernoulli utility implies that, once the corresponding welfare information has been fixed, the measures RIRA_t and AIRA_t are determined independently of the representation and its gauge. In assertion a) this welfare information corresponds to fixing the unit of measurement, by prescribing a numerical value to the difference in welfare between the best and the worst outcome, i.e. $u_1^{\text{welf}}(x^{\max}) - u_1^{\text{welf}}(x^{\min}) = g \circ u(x^{\max}) - g \circ u(x^{\min}) = \overline{G} - \underline{G} = w^*$. Such a partial specification of the measure scale for welfare makes the measures of *absolute* intertemporal risk aversion unique. Assertion b) fixes the 'zero welfare level', by choosing an outcome that shall correspond to zero welfare. The information is enough to render the measures of *relative* intertemporal risk aversion unique. Assertion c) fixes the welfare unit and the zero welfare level together. This step completely eliminates the freedom in the choice of measure scale for welfare. In consequence, both measures of intertemporal risk aversion are determined uniquely. Assertion d) offers an alternative way to eliminate the indeterminacy of the measure scale for welfare, by specifying the range of the function g and, thus, the welfare levels corresponding to the best and the worst outcomes. The latter approach is taken in the subsequent corollaries.

For preferences satisfying risk stationarity II as formulated in axiom A7, it is worthwhile to take a closer look at the representations that fix the degree of freedom in the measure scale for welfare. For this purpose define the uncertainty aggregation rule $\mathcal{M}^{\exp^\xi}$ for the case $\xi = 0$ by limit, yielding[18]

$$\mathcal{M}^{\exp^0}(p_t, \tilde{w}_t) \equiv \lim_{\xi \to 0} \mathcal{M}^{\exp^\xi}(p_t, \tilde{w}_t) = \lim_{\xi \to 0} \frac{1}{\xi} \ln \left[\int dp_t \exp(\xi \tilde{w}_t) \right] = E_{p_t} \tilde{w}_t \,.$$

The limit is a simple application of l'Hospital's rule, as shown in the proof of corollary 8. Gauging g to identity in the representation of theorem 9 and fixing the range of welfare, which for $g = \text{id}$ corresponds to the range of u, I obtain the following representation.

Corollary 8 **($g = \text{id}^+$−gauge)** : Choose a nondegenerate closed interval $W^* \subset \mathbb{R}$.

[18]Note that the characterization of the uncertainty aggregation rule by $f = \exp^\xi$ is equivalent to $f(z) = \exp(\xi\, z)$.

A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1,...,T\}$ (vNM setting)

ii) A4' for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) A9 (risk stationarity II)

if and only if, there exists a continuous and surjective function $u : X \to W^*$, a discount factor $\beta \in \mathrm{I\!R}_{++}$ and $\xi \in \mathrm{I\!R}$, such that with defining the functions

v) $\tilde{w}_t : \tilde{X}_t \to \mathrm{I\!R}$ for $t \in \{1,...,T\}$ by $\tilde{w}_T(x_T) = u(x_T)$ and recursively

$$\tilde{w}_{t-1}(x_{t-1}, p_t) = u(x_{t-1}) + \beta\, \mathcal{M}^{\exp^\xi}(p_t, \tilde{w}_t) \tag{9.11}$$

it holds for all $t \in \{1,...,T\}$ that

$$p_t \succeq_t p'_t \quad \Leftrightarrow \quad \mathcal{M}^{\exp^\xi}(p'_t, \tilde{w}_t) \geq \mathcal{M}^{\exp^\xi}(p'_t, \tilde{w}_t) \quad \forall\, p_t, p'_t \in P_t. \tag{9.12}$$

Moreover, the function u is determined uniquely. With the convention that $g_1 = g$, the uniquely defined measures of intertemporal risk aversion are calculated to $\text{AIRA}_t = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}$ and $\text{RIRA}_t = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}\,\text{id}$.

Fixing the welfare range eliminates the affine freedom of g, here corresponding to the freedom of u ($g = \text{id}-$gauge). In the representation of theorem 9, part of this freedom was employed to carry information over the relation between the functions g and f_t. Fixing g and its range exogenously, this information gives rise to the new parameter ξ. It parametrizes intertemporal risk aversion and corresponds to a degree of freedom between the function g, characterizing intertemporal aggregation, and the functions f_t, characterizing uncertainty aggregation. In the particular case where $\xi = 0$, the coefficient of relative atemporal risk aversion (defined in a one commodity setting) is confined to the inverse of the intertemporal elasticity of substitution and intertemporal risk neutrality prevails. In this case, equations (9.11) and (9.12) recursively define the intertemporally additive expected utility framework.

In theorem 9 there appear three seemingly disconnected representations corresponding to $h \in \left\{\frac{1}{\exp}, \text{id}, \exp\right\}$. Corollary 8 shows how the coefficient of absolute intertemporal risk aversion, which is proportional to ξ, connects the three different classes continuously, allowing for a wide range of intertemporal risk attitude. However, though being constant in welfare, the coefficient of absolute intertemporal risk aversion is not constant over time. In the discussion of theorem 10, it had already been observed that for risk stationary representations in the sense of axiom A8 only the expression $f_t \circ g^{-1}$, which I referred to as the stationary characterization of intertemporal risk aversion, stays constant over time. The general characterization $f_t \circ g_t^{-1}$ picks up the discount rate from $g_t = \beta^{t-1} g$. The same happens for risk stationarity II in the sense of axiom A9. The interpretation is as follows. The function $f_t \circ g^{-1}$ characterizes intertemporal risk aversion in period t with respect to a welfare scale that is fixed in period t to $\text{range}(g) = \text{range}(u) = W^*$. One could formulate this characterization as a measurement in terms of a 'current value measure scale for welfare'. With respect to such a constant 'current value measure scale', the characterizing functions of intertemporal risk aversion are constant over time.[19] In contrast, the measures AIRA_t and RIRA_t are defined with respect to the characterizing functions $f_t \circ g_t^{-1}$. Fixing $\text{range}(g_1) = \text{range}(g) = \text{range}(u) = W^*$, implies that $\text{range}(g_t) = \beta^{t-1}\text{range}(g) = \beta^{t-1} W^*$. Thus, in these measures intertemporal risk aversion is measured with respect to a 'present value measure scale for welfare' and discounting shrinks the range of welfare that serves as basis for the measurement of intertemporal risk aversion in period t. Then, as the range of the welfare measure scale (in present value) becomes smaller and smaller over time due to discounting, the coefficient of intertemporal risk aversion has to increase in order to keep up a stationary aversion to risk. Therefore, the coefficients of intertemporal risk aversion AIRA_t and RIRA_t include the factor β^t in the denominator. In addition to the latter, risk stationarity II brings another dependence on time into the characterization of intertemporal risk aversion. In the denominator appears as well the time-dependent normalized discount factor β_t. Recall that the latter takes account of the relative weight given to a single period as opposed to the remaining future, a weight changing over time when a finite planning horizon is approached. For a representation satisfying risk stationarity in the sense of axiom A9, this change of weight enters into the characterization of intertemporal risk aversion. It implies that the stationary part of intertemporal risk aversion, characterized by $f_t \circ g^{-1}$, slowly decreases over time as the term $1 - \beta_t$ increases to unity for the last period. Leaving this term unconsidered, yields a representation in the sense of theorem 8, satisfying axiom A8. In other words, *disregarding the adjustment of intertemporal risk aversion by the change of weight that the remaining future obtains as opposed to the present period*, in a setting with a finite planning horizon, *makes the corresponding decision maker indifferent to the length of risk taking (axiom A8).* For an infinite time horizon this weight is obviously constant, precisely it holds $(1-\beta_t) = (1-\beta)$, and a representation in the (limiting) sense of theorem 9 also is a representation in the (limiting)

[19]Except for the normalization factor $\frac{1}{1-\beta_t}$ in the case of risk stationarity in the sense of axiom A9. This term will be discussed further below.

sense of theorem 8.[20]

Alternatively to corollary 8, the representation can also be stated in the Kreps and Porteus gauge where the functions f_t are set to identity.

Corollary 9 **($f = \mathbf{id}^+$-gauge)** : Choose a nondegenerate closed interval $U^* \subset \mathrm{I\!R}_{++}$. A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting)

ii) A4 for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) A9 (risk stationarity II)

if and only if, there exists a continuous and surjective function $u : X \to U^*$, a discount factor $\beta \in \mathrm{I\!R}_{++}$ and $\xi \in \mathrm{I\!R}$, such that defining the functions

v) $\tilde{u}_t : \tilde{X}_t \to \mathrm{I\!R}$ for $t \in \{1, ..., T\}$ by $\tilde{u}_T(x_T) = u(x_T)$ and recursively

- for $\xi > 0$: $\tilde{u}_{t-1}(x_{t-1}, p_t) = u(x_{t-1})^\xi \big(\mathrm{E}_{p_t} \tilde{u}_t\big)^\beta$ and
- for $\xi = 0$: $\tilde{u}_{t-1}(x_{t-1}, p_t) = \ln u(x_{t-1}) + \beta\, \mathrm{E}_{p_t} \tilde{u}_t$ and
- for $\xi < 0$: $\tilde{u}_{t-1}(x_{t-1}, p_t) = -u(x_{t-1})^\xi \big(-\mathrm{E}_{p_t} \tilde{u}_t\big)^\beta$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p'_t \quad \Leftrightarrow \quad \mathrm{E}_{p_t} \tilde{u}_t \geq \mathrm{E}_{p'_t} \tilde{u}_t \quad \forall\, p_t, p'_t \in P_t\,.$$

Moreover, the function u is determined uniquely. With the convention that $g_1 = g$,[21] the uniquely defined measures of intertemporal risk aversion are calculated to $\mathrm{AIRA}_t = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}$ and $\mathrm{RIRA}_t = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}\,\mathrm{id}$.

Here, the functions $\tilde{u}_t$ are the same as those used in representation theorem 4. The function u, however, is only a strictly monotonic transformation of the respective Bernoulli utility function used in the latter theorem. It is chosen in such a way that fixing the range of u also fixes the range of g,[22] which is necessary to render the measures of risk aversion unique. However, note that due to the multiplicative form of intertemporal

[20] Note that a constant term $(1-\beta)$ can be absorbed into the parameter ξ and makes no difference for the comparison between different classes of representations.

[21] This notation relates to the underlying representing triples in the sense of theorem 4. In corollary 9 the assumption implies that the measure scale of welfare is fixed for the first period to $\mathrm{range}(u_1^{\mathrm{welf}}) = \mathrm{range}(g \circ u) = \ln U^*$. See also the discussion below and the first remark in the proof of corollary 9.

[22] This is possible because of the relation implied by axiom A9 between the functions f_t and g.

aggregation, the range of welfare u_1^{welf} in the certainty additive sense of chapter 7.3 is fixed to the range $W^* = \ln U^*$ rather than to the range U^*. This is also the reason, why a logarithm appears in the representation for $\xi = 0$. Only with this definition, the range of welfare is fixed independently of the parameter ξ. However, eliminating the logarithm in the representation for $\xi = 0$ would not change the measures of intertemporal risk aversion, as they are zero in the case $\xi = 0$ and, thus, independent of the particular measure scale adopted for welfare.

Observe the particular nonlinear form for intertemporal aggregation that arises when uncertainty aggregation is required to be linear. For decision makers that are not intertemporally risk neutral, it is 'almost multiplicative'. But it depends on the exponent ξ. Translating the exponent ξ back into the uncertainty aggregation rule and establishing a purely multiplicative intertemporal aggregation yields the following representation.

Corollary 10 (isoelastic uncertainty evaluation): Choose a nondegenerate closed interval $U^* \subset \mathbb{R}_{++}$. A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting)

ii) A4' for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) A9 (risk stationarity II)

if and only if, there exists a continuous and surjective function $u : X \to U^*$, a discount factor $\beta \in \mathbb{R}_{++}$ and $\xi \in \mathbb{R}$, such that defining the functions

v) $\tilde{v}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by $\tilde{v}_T(x_T) = u(x_T)$ and recursively

$$\tilde{v}_{t-1}(x_{t-1}, p_t) = u(x_{t-1}) \left(\mathcal{M}^{\alpha=\xi}(p_t, \tilde{v}_t)\right)^{\beta} \tag{9.13}$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p'_t \quad \Leftrightarrow \quad \mathcal{M}^{\alpha=\xi}(p'_t, \tilde{v}_t) \geq \mathcal{M}^{\alpha=\xi}(p'_t, \tilde{v}_t) \quad \forall\, p_t, p'_t \in P_t.$$

Moreover, the function u is determined uniquely. With the convention that $g_1 = g$,[23] the uniquely defined measures of intertemporal risk aversion are calculated to $\text{AIRA}_t = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}$ and $\text{RIRA}_t = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}$ id.

[23] See footnote 21.

Note, that the recursive construction (9.13) of the representation for $\xi = 0$ is equivalent to the intertemporally additive expected utility setting. Again the range of welfare in the sense of chapter 7.3 corresponds to $W^* = \ln U^*$. The above representation is particularly interesting, because it points out a special case that closely relates to the generalized isoelastic framework analyzed in section 9.2. In the one commodity setting and for $u = \text{id}$ the following recursive characterization of the decision maker's evaluation is obtained:

$$\tilde{v}_{t-1}(x_{t-1}, p_t) = x_{t-1} \left(\mathcal{M}^{\alpha=\xi}(p_t, \tilde{v}_t)\right)^{\beta} . \tag{9.14}$$

It corresponds to an intertemporal elasticity of substitution of unity ($\rho = 0$) and uses the isoelastic uncertainty aggregation rule $\mathcal{M}^\alpha$. Taking the interpretation of section 7.3, the case $\rho = 0$, i.e. $g = \ln$, corresponds to logarithmic welfare, which is a widespread assumption in macroeconomics and popular also in environmental economic modeling. It is the only specification for which risk stationarity II in the sense of axiom A9 allows an isoelastic uncertainty aggregation.[24] Observe that the setting (9.14) only coincides with the (corresponding special case of the) isoelastic representation of section 9.2 for an infinite planning horizon. As pointed out on page 144, the representations in the sense of theorems 8 and 9 differ in the way they take account of the approaching end of the planning horizon. More precisely, only the representation based on axiom A9 incorporates the change in weight that the present receives as opposed to the remaining future 'which shortens over time'. Let me summarize that only risk stationarity in the sense of axiom A9 is a proper translation of the assumption that the mere passage of time does not affect preferences. In contrast, axiom A8 characterizes what it needs to make atemporal uncertainty aggregation constant in a setting with a finite time horizon. This condition can be expressed as a form of indifference to the length of risk taking.

[24] In order to render f_t a power function, intertemporal risk aversion has to be of the form $f_t \circ g^{-1} = \left(g^{-1}\right)^\alpha$. This expression can only be proportional to $\exp^\xi$, characterizing up to proportionality intertemporal risk aversion in the representation of theorem 9, if g^{-1} is proportional to exp. But then g has to be proportional to ln which corresponds to the case $\rho = 0$.

Chapter 10

Temporal Resolution of Uncertainty

10.1 A Preference for the Timing of Uncertainty Resolution

A particular feature of the recursive utility models employed in the preceding chapters, is that they allow for an intrinsic preference for early or late resolution of uncertainty. This preference is intrinsic in the sense that a decision maker can strictly prefer an early resolution of uncertainty, even if the information obtained from the early resolution is known not to affect his plans and, thus, his future outcomes. The current section analyzes how a preference for early or late resolution of uncertainty is expressed in the representations derived in the preceding chapters. The corresponding theorem is a straight forward adaption of the result obtained by Kreps & Porteus (1978) to my gaugeable setting. However, expressing the condition in terms of intertemporal risk aversion, allows to examine the reason *why* a decision maker in the recursive utility framework can exhibit a preference for the timing of uncertainty resolution. Section 10.2 discusses why such a form of intrinsic timing preference might not be desired in a principled approach to choice under uncertainty. A consequence of eliminating the intrinsic timing preference from the model is that, instead of recursive temporal lotteries, the more common description of uncertainty through probability measures that are directly defined on consumption paths becomes sufficient for the evaluation of the uncertain future. I elaborate, how the latter standard measures are derived from a given temporal lottery by 'integrating out' temporal information. Subsequently, section 10.3 states the preference representation for a decision maker who is indifferent with respect to the timing of uncertainty resolution. Similarly to chapter 9.3, the resulting representation captures intertemporal risk aversion in a single parameter. I work out that the resulting model allows to disentangle (atemporal) risk aversion from intertemporal substitutability without employing temporal lotteries. Furthermore, I show that a particularly suggestive ordering of two lotteries, which is used in the literature to motivate a non-trivial preference with respect to the timing of uncertainty resolution, can be satisfied as well under indifference to the timing of uncertainty resolution, if the decision maker exhibits intertemporal risk aversion. Section 10.4 elaborates implications of the axioms worked out in this and the preceding chapter for the pure rate of time preference and, more generally, the weight given to future welfare.

Kreps & Porteus (1978) show that their recursive approach to describe choice under uncertainty, which I have adopted so far, allows a decision maker's preference to depend on the time at which uncertainty resolves. Such a characteristic of preferences differs from an instrumental preference for an early resolution of uncertainty, which is also possible in the intertemporally additive expected utility model. In the latter case, early resolution of uncertainty is preferred, whenever the information stemming from an earlier resolution of uncertainty can be used to take action that improves outcomes (or their probabilities). In contrast, the preference for early resolution of uncertainty that is allowed in the setup of Kreps & Porteus (1978) and, thus, my setup in chapters 8 and 9, even holds when the decision maker cannot make any use of the information. Chew & Epstein (1989) further analyze such a preference for the timing of uncertainty resolution by introducing the concept of a timing premium for an early resolution. In particular, they derive a representation for preferences going along with a constant timing premium.[1] In fact, the authors show that the assumption of a constant timing premium can replace the independence axiom, which I adopt throughout my analysis (Chew & Epstein 1989, 110). Note that any nontrivial[2] application of the generalized isoelastic model discussed in the preceding chapter implicitly features a preference for the timing of uncertainty resolution. The explicit analysis of such a timing preference is carried over to a time-continuous setting by Duffie, Schroder & Skiadis's (1997), who analyze how it influences asset pricing. Further generalizations of the modeling

[1] The timing premium relates the probabilities of two lotteries, in which uncertainty resolves at different points of time, and that yield the same outcomes with different probabilities. These probabilities are picked in a way to make the decision maker indifferent between the lottery featuring early and the lottery featuring late resolution of uncertainty.

[2] That is, any setting with non-zero consumption where the generalized model differs from intertemporally additive expected utility and the coefficient of relative risk aversion does not coincide with the inverse of the elasticity of intertemporal substitutuion.

framework are treated in Grant, Kajii & Polak (2000) and Skiadas (1998), who generalize the concept of a comparable intrinsic preference for the timing of uncertainty resolution beyond a recursive setup on temporal lotteries.

A standard motivation in the literature on non-indifference with respect to the timing of uncertainty resolution is based on the comparison of the following two lotteries (e.g. Duffie & Epstein 1992, Duffie et al. 1997, Skiadas 1998). In both lotteries, a decision maker faces for some fixed number of periods either a high or a low consumption level, depending on the toss of a coin. In lottery A, the coin is tossed at the beginning of *every* period. If head comes up, the decision maker receives the high payoff *in the respective period* and, if tail comes up, the decision maker faces the low payoff. In lottery B, a coin is tossed *once* at the beginning of the first period. If head comes up, the agent receives the high payoff in *all* periods, if tail comes up the agent receives the inferior payoff in *all* periods. It is easily verified that a decision maker who employs the intertemporally additive expected utility model, is indifferent between the lotteries A and B (see section 10.3). The intuition appealed to in the literature, which I personally share, is that people would usually prefer lottery A over lottery B. Since the coin in lottery A is flipped in every period, while in lottery B all uncertainty is resolved in the first period, such a lottery evaluation can be interpreted as a preference for a late resolution of uncertainty. However, the perfect serial correlation of outcomes in lottery B and the independence of the outcomes in lottery A depicts another important difference between the two lotteries. In fact, in section 10.3 I show that a strict preference for lottery A can also be derived in a non-recursive model, where the decision maker is indifferent with respect to the timing of uncertainty resolution, but intertemporally risk averse. The following extension of the lottery example, found in the same papers cited above, concentrates on a pure timing preference. For this purpose, a third lottery C is introduced. It coincides with lottery A, except for the fact that for every period coins are (independently) tossed at the beginning of period one. Therefore, the uncertainty resolves at the same point as in lottery B, i.e. early. Again the authors point out that lotteries A and C conceptually differ, and that a decision maker might not be indifferent between tossing the coins in the first period (lottery C) or in the respective periods (lottery A) “based on the psychic costs and benefits of early resolution” (Duffie & Epstein 1992). Other situations appealed to in the literature, where people might exhibit intrinsic preference for either late or early resolution of uncertainty, include anxious PhD students receiving information on exams or jobmarket placements before or after a vacation (Chew & Epstein 1989, Grant, Kajii & Polak 1998), or a person facing information on an incurable genetic disorder (Grant et al. 1998).

In the following I introduce the precise definition of an intrinsic[3] preference for the timing of uncertainty resolution. Let $\lambda(x_t, p_{t+1}) + (1-\lambda)(x_t, p'_{t+1})$ denote a lottery in period t that yields (x_t, p_{t+1}) with probability λ and (x_t, p'_{t+1}) otherwise. Both outcomes of the 'λ-lottery' yield the same consumption x_t in period t. Only the outcomes from period $t+1$ on, described by p_{t+1} and p'_{t+1}, are allowed to differ. An individual facing the above lottery will know at the end of period t whether he confronts p_{t+1} at the beginning of period $t+1$ or whether he faces p'_{t+1}. Thus, uncertainty over p_{t+1} versus p'_{t+1} resolves in period t. The definition of an intrinsic preference for the timing of uncertainty resolution compares the lottery above to its degenerate counterpart $(x_t, \lambda p_{t+1} + (1-\lambda)p'_{t+1})$. Analogous to the first lottery, the latter yields x_t with certainty, p_{t+1} with probability λ, and p'_{t+1} with probability $1-\lambda$. However, the uncertainty about facing p_{t+1} or p'_{t+1} in period $t+1$ is not resolved in period t, but only in period $t+1$. The following definition applies.

Definition: A decision maker *prefers early [late] resolution of uncertainty* in period t for the fixed outcome x_t, if for all $p_{t+1}, p'_{t+1} \in P_{t+1}, \lambda \in [0,1]$ it holds that

$$\lambda(x_t, p_{t+1}) + (1-\lambda)(x_t, p'_{t+1}) \quad \succeq_t \quad [\preceq_t] \quad (x_t, \lambda p_{t+1} + (1-\lambda)p'_{t+1}). \tag{10.1}$$

In words, a decision maker with a preference for early resolution of uncertainty, prefers when the uncertainty about facing the future described by p_{t+1} or the future described by p'_{t+1} resolves already in period t (lottery on the left), rather than in period $t+1$ (lottery on the right). This uncertainty corresponds to the probability mixture λ and $1-\lambda$ in equation (10.1). In contrast, if the decision maker exhibits a preference for late resolution of uncertainty, he prefers to keep the uncertainty in period t, and have it resolved only in period $t+1$. Note that a decision maker's attitude with respect to the timing of uncertainty resolution can generally depend on the period t, as well as on the (certain) outcome x_t, which he is facing in the respective period. A decision maker is called *indifferent* to the timing of uncertainty resolution, if equation (10.1) holds with indifference ($\sim_t$ replacing $\succeq_t$).

For a decision maker subscribing to the non-stationary multiperiod setting of chapter 8, his attitude with respect to the timing of uncertainty resolution can be characterized in terms of the representation of theorem 4 as follows.

[3] Note that the word '*intrinsic*' is not standard in the corresponding literature. I introduce the word to stress the conceptual *difference to* an *instrumental* preference for an early resolution of uncertainty, which is implied by a possibility to use the early arrival of information in order to raise (expected) welfare.

Theorem 11: Let the sequence $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ represent the preferences $(\succeq)_{t\in\{1,...,T\}}$ in the sense of theorem 4. Then a decision maker prefers early [late] resolution of uncertainty in period t for outcome x_t, if and only if,

$$f_t \circ g_t^{-1}\left[\theta_t\, g_t \circ u_t(x_t) + \theta_t \theta_{t+1}^{-1}\, g_{t+1} \circ f_{t+1}^{-1}(z) + \theta_t \theta_{t+1}^{-1} \vartheta_t\right] \tag{10.2}$$

is convex [concave] in $z \in f_t(U_t)$.[4]

The theorem adapts the result of Kreps & Porteus (1978) to the certainty separable framework, and extends it in the sense of allowing for general gauges of intertemporal and uncertainty aggregation. At the same time, my version relates the preference for the timing of uncertainty resolution to the concept of intertemporal risk aversion. An immediate consequence of theorem 11 is that *only an intertemporally risk averse or risk seeking decision maker can exhibit a (strict) preference for early or late resolution of uncertainty*. Otherwise, if $f_t \circ g_t^{-1}$ and $f_{t+1} \circ g_{t+1}^{-1}$ are linear, expression (10.2) is linear as well. In such a situation the decision maker is indifferent with respect to the timing of uncertainty resolution. As section 10.3 elaborates, the opposite is not true, i.e. a decision maker who is indifferent to the timing of uncertainty resolution is not necessarily intertemporally risk neutral. Note that, comparing my result to Kreps & Porteus (1978, 197), doing without additive separability on certain consumption paths implies replacing my expression in equation (10.2) by a general function $\tilde{u}_t(x_t, z)$.[5]

Theorem 11 answers a question raised by Epstein & Zin (1989, 952 et seq.) on the interlacement of the preference for early resolution of uncertainty with risk aversion and intertemporal substitutability. In their u = id-gauge, Epstein & Zin (1989) find for the stationary isoelastic setting that early resolution of uncertainty is preferred, if and only if $\alpha < \rho$.[6] The authors pose the question how these three characteristics of

[4]If theorem 4 is stated in terms of strictly monotonic functions (see chapter 8, footnote 2), then the latter part changes to: '...if and only if (10.2) is convex [concave] in $z \in f(U)$ for an increasing version of f, or concave [convex] for a decreasing choice of f'.

[5]Kreps & Porteus's (1978) expression $\tilde{u}_t(x_t, z)$ corresponds to the f = id gauge. Moreover, comparing my theorem 11 to their theorem 3, note that my expression (10.2) is always increasing in z, which has to be assured for a representation in the sense of Kreps & Porteus's (1978) theorem 1.

[6]Using theorem 11, the result is verified easily for $x > 0$. For $x = 0$ the decision maker is always indifferent with respect to the resolution of uncertainty in the respective period (see footnote 10). In the generalized isoelastic setting expression (10.2) turns into $\left[x_t^\rho + \beta_t\, z^{\frac{\rho}{\alpha}}\right]^{\frac{\alpha}{\rho}}$. Calculating the second order derivative in z reproduces Epstein & Zin's (1989, 952) result. Precisely, for $\alpha > 0$ the second order derivative is strictly positive, if and only if $\alpha < \rho$. For the case $\alpha < 0$ the function $f_t = z^\alpha$ is decreasing and, thus, the convexity condition for the attitude with respect to the timing of uncertainty resolution reverses (see footnote 4). In that case, the second order derivative turns out strictly negative, if and only if $\alpha < \rho$. For the cases where α or ρ equal zero, set f respectively g to the logarithm.

preference are related to each other in a more general setting.[7] In a one commodity setting with utility strictly increasing in the consumption level, expression (10.2) brings an answer by gauging u_t to identity. Then, the functions f_t and g_t characterize general uncertainty attitude and intertemporal substitutability, and theorem 11 states their exact relation to the preference for an early resolution of uncertainty. In particular, it can be observed that the most important determinant of the preference for early or late resolution of uncertainty is the difference $f_t \circ g_t^{-1}$ between the functions characterizing (atemporal) risk aversion and intertemporal substitutability in two adjacent periods. Having elaborated the interpretation of the term $f_t \circ g_t^{-1}$ as characterizing intertemporal risk aversion, the latter concept should also foster understanding *why* a decision maker in a Kreps & Porteus (1978) framework exhibits an intrinsic preference for early or late resolution of uncertainty.

The following scenario should help to understand, why $\alpha < \rho$ in the isoelasitc setting of Epstein & Zin (1989) and convexity of the expression (10.2) in the general setting, describe a decision maker with a preference for an early resolution of uncertainty. In a two period setup, a decision maker faces two different lotteries. Both lotteries yield with equal probability either a high or a low outcome in the second period, and a common certain outcome in the first. However, in lottery E (*e*arly), which is formally defined as $\frac{1}{2}(x_1, \overline{x}_2) + \frac{1}{2}(x_1, \underline{x}_2)$, the uncertainty over the second period resolves already in period 1. On the contrary, in lottery L (*l*ate), which is defined as $(x_1, \frac{1}{2}\overline{x}_2 + \frac{1}{2}\underline{x}_2)$, the uncertainty resolves in period 2. To simplify the notation, let p be the probability measure giving weight $p_1 = p_2 = \frac{1}{2}$ to outcomes $x_2^1 = \overline{x}_2$ and $x_2^2 = \underline{x}_2$. Moreover, let $i \in \{1, 2\}$ and, thus, $x_2^i \in \{\overline{x}_2, \underline{x}_2\}$. Assuming a certainty stationary decision maker who subscribes to the axioms of representation theorem 7, the decision maker prefers lottery E over lottery L, if and only if the following relation holds:

$$
\begin{aligned}
&\frac{1}{2}(x_1, \overline{x}_2) + \frac{1}{2}(x_1, \underline{x}_2) \;\; \succeq_1 \;\; (x_1, \frac{1}{2}\overline{x}_2 + \frac{1}{2}\underline{x}_2) \\
\Leftrightarrow\; & f_1^{-1}\Big\{ \textstyle\sum_i p_i f_1 \circ g^{-1}\Big[(1-\beta_1)\, g \circ u(x_1) + \beta_1\, g \circ u(x_2^i)\Big]\Big\} \\
&\quad \geq g^{-1}\Big[(1-\beta_1)\, g \circ u(x_1) + \beta_1\, g \circ f_2\Big\{ \textstyle\sum_i p_i f_2 \circ g^{-1} \circ g \circ u(x_2^i)\Big\}\Big]
\end{aligned}
$$

[7]Epstein & Zin (1989, 952): "For more general recursive utility functions, we have not found a characterization in terms of [time and uncertainty aggregators] of the condition under which early or late resolution is preferred" and they "suspect [...] inherent inseparability of these three aspects [...] Further study of this issue is required" (Epstein & Zin 1989, 953).

$$\Leftrightarrow g \circ f_1^{-1}\Big\{ \textstyle\sum_i p_i f_1 \circ g^{-1}\Big[(1-\beta_1)\, g \circ u(x_1) + \beta_1\, g \circ u(x_2^i)\Big]\Big\}$$
$$\geq (1-\beta_1)\, g \circ u(x_1) + \beta_1\, g \circ f_2^{-1}\Big\{ \textstyle\sum_i p_i f_2 \circ g^{-1} \circ g \circ u(x_2^i)\Big\}$$
$$\Leftrightarrow \mathcal{M}^{f_1 \circ g^{-1}}\Big(p, (1-\beta_1)\, g \circ u(x_1) + \beta_1\, g \circ u\Big) \tag{10.3}$$
$$\geq (1-\beta_1)\, g \circ u(x_1) + \beta_1\, \mathcal{M}^{f_2 \circ g^{-1}}\Big(p, g \circ u\Big)\,. \tag{10.4}$$

The simplest but very important insight is that the recursive framework evaluates the risk in the period in which it resolves. Expression (10.3) can be interpreted as an evaluation of lottery E, featuring an early resolution of uncertainty. The uncertainty is evaluated with an uncertainty aggregation rule that is characterized by intertemporal risk aversion in period 1.[8] Expression (10.4) corresponds to the evaluation of lottery L, featuring a late resolution of uncertainty. Here, intertemporal risk aversion in period 2 is used to evaluate the corresponding risk. Thus, both lottery evaluations can be expressed by means of an uncertainty aggregation rule which is characterized by intertemporal risk aversion. *However, the difference in the evaluation is twofold. First*, to evaluate *lottery E* the decision maker bases his evaluation of uncertainty on *intertemporal risk aversion in period* 1, characterized by $f_1 \circ g^{-1}$. For an evaluation of *lottery L*, however, he applies the uncertainty aggregation rule $\mathcal{M}^{f_2 \circ g^{-1}}$, which is characterized by *intertemporal risk aversion as it holds in period* 2. *Second*, the fact that uncertainty resolves for *lottery E* in the first period, makes the *'whole' consumption path* from period 1 to period 2 an argument of the uncertainty aggregation rule. This *includes* the *certain consumption in period* 1, as well as the *discount factor* β_1. In contrast, in the evaluation of *lottery L*, the uncertainty aggregation rule is *only applied to the outcomes of the second period.*

Still striving for an intuition why a decision maker might intrinsically prefer an early resolution of uncertainty, let me further simplify the scenario. Assume, as it is done in Epstein & Zin (1989), that the functions $f_1 = f_2 \equiv f$ coincide for both periods. Moreover, assume that the first period outcome x_1 corresponds to the decision maker's zero welfare level.[9] Then the above inequality, corresponding to a preference for the lottery with the early resolution of uncertainty, writes as

$$\mathcal{M}^{f \circ g^{-1}}\Big(p, \beta_1\, g \circ u\Big) \geq \beta_1\, \mathcal{M}^{f \circ g^{-1}}\Big(p, g \circ u\Big)\,. \tag{10.5}$$

---

[8] To arrive at the evaluation in the last equivalence, only a strictly increasing transformation has been applied to both sides of the inequality. Therefore, both sides still can be interpreted as an evaluation of the respective lottery.

[9] This is a zero welfare level in the sense of chapter 7.3, i.e. $u^{\text{welf}} = g \circ u(x_1) = 0$. For the isoelastic one commodity Epstein Zin setting it implies a zero consumption level $x = 0$.

The only difference left between the two lottery evaluations, is that for the lottery with an early resolution of uncertainty (left), the decision maker's devaluation of the future finds its place *in* the argument of the uncertainty aggregation rule. This position of β_1 is due to the fact that lottery E is conceived as a lottery over the whole consumption path, and the weight given to the future is part of the evaluation of the path. On the other hand, for lottery L, the uncertainty is related directly to the second period outcomes. Only after the corresponding uncertainty is evaluated, the resulting certainty equivalent is discounted. In such a scenario, the generalized isoelastic model yields a particular evaluation as the uncertainty aggregation rules are linear homogeneous:

$$\begin{aligned}\mathcal{M}^{f\circ g^{-1}}(p, \beta_1\, g\circ u) &= \left[\sum_i p_i\, (\beta_1 x_2^i)^{\frac{\alpha}{\rho}}\right]^{\frac{\rho}{\alpha}} = \beta_1 \left[\sum_i p_i\, (x_2^i)^{\frac{\alpha}{\rho}}\right]^{\frac{\rho}{\alpha}} \\ &= \beta_1\, \mathcal{M}^{f\circ g^{-1}}(p, g\circ u)\ .\end{aligned}$$

In consequence, for a zero consumption level in the first period, the decision maker is indifferent between the two lotteries.[10] Observe, that this indifference whether or not the normalized discount factor β_1 is included in the lottery evaluation, only holds for linear homogeneous uncertainty aggregation rules. In general, a devaluation of the future makes a difference in the recursive evaluation of the two lotteries. For $\beta_1 < 1$, the inequality (10.5) is strict, if the uncertainty aggregation rule exhibits 'decreasing returns to scale' in its second argument.[11] Decreasing returns imply that discounting the certainty equivalent of the undiscounted period 2 lottery, yields a lower evaluation, than applying the uncertainty aggregation rule to the discounted outcomes. The first procedure is taken to evaluate lottery L with a late resolution of uncertainty, the latter to evaluate lottery E exhibiting an early resolution of uncertainty. Summarizing in one sentence, *if the decision maker's uncertainty evaluation of discounted lottery outcomes is higher than his discounted uncertainty evaluation of the undiscounted lottery outcomes, he prefers an early resolution of uncertainty.*

Now consider an evaluation of the above lotteries when the first period outcome does not correspond to a zero welfare level. The uncertainty aggregation in the evaluation of lottery L still yields the same result. The evaluation of the certain first period outcome is added only after the evaluation of the uncertain second period. In contrast, uncertainty

[10]This special case has not been pointed out by Epstein & Zin (1989). To verify it, use theorem 11. In the generalized isoelastic setting, expression (10.2) turns into $\left[x_t^{\rho} + \beta_t\, z^{\frac{\rho}{\alpha}}\right]^{\frac{\alpha}{\rho}}$. It is immediate that for $x_t = 0$ the expression is linear in z.

[11]Note that the normalized discount factor b_1 is smaller than one. Therefore, the inequality in the definition of decreasing returns to scale is inverted compared to a definition of decreasing returns to scale with the help of some $\lambda > 1$.

aggregation in lottery E is no longer concerned only with the second period outcomes x_2^i. Also outcome x_1 is conceived as part of the lottery entering the uncertainty evaluation. Thus, introducing a positive first period welfare level raises the mean outcome of lottery E, while keeping the variability in the lottery identical to the above setting with a zero welfare level in the first period. In the Epstein-Zin setting with generalized isoelastic preferences, the consequences are as follows. Recall, that in the isoelastic setting the decision maker exhibits a constant coefficient of relative intertemporal risk aversion (parametrized by $\frac{\alpha}{\rho}$). Thus, in absolute terms, the decision maker is less risk averse when evaluating a lottery that implies a particular absolute variability of outcomes at a higher welfare level, than he is when evaluation a lottery with the same variability of outcomes at a lower welfare level. Therefore, he, effectively, is less risk averse when evaluating the lottery with an early resolution of uncertainty, than he is when evaluating the lottery with a late resolution of uncertainty (in the latter lottery evaluation the first period consumption does not enter the uncertainty aggregation rule). In consequence, *a decision maker who is intertemporally risk averse and exhibits a constant coefficient of relative intertemporal risk aversion, prefers an early resolution of uncertainty.* On the other hand, a decision maker who is intertemporally risk averse but exhibits a constant coefficient of *absolute* intertemporal risk aversion does not necessarily prefer an early resolution of uncertainty. This will be derived formally in section 10.3. *In general*, this second effect fostering a preference for an early resolution of uncertainty, is based on the model feature that *an earlier resolution of uncertainty makes the decision maker evaluate the constant welfare spread between the uncertain outcomes at a higher welfare level*, whenever the foregoing certain period yields positive welfare.

A third effect causing a preference for an early resolution of uncertainty can set in when the assumption that $f_1 = f_2$ is relaxed. Assume that the decision maker exhibits a *stronger intertemporal risk aversion in period* 2*, than* he does *in period* 1. Then, abstracting from the other effects worked out above, uncertainty reduces the welfare stronger when the decision maker evaluates a lottery resolving in period 2, i.e. late. This is, because in the latter case he applies the uncertainty aggregation rule $\mathcal{M}^{f_2 \circ g^{-1}}$ to evaluate the uncertain outcomes, while for the evaluation of the lottery with an early resolution of uncertainty, the decision maker employs the less risk averse uncertainty aggregation rule $\mathcal{M}^{f_1 \circ g^{-1}}$.

The three possible driving forces for a preference for an early resolution of uncertainty identified above, are merged in expression (10.2) of theorem 11. Starting with the one treated last, it appears in terms of the functions $f_t \circ g_t^{-1}$ and $\left[f_{t+1} \circ g_{t+1}^{-1}\right]^{-1}$. Consider the case where $f_t \circ g_t^{-1}$ is linear, while $f_{t+1} \circ g_{t+1}^{-1}$ is concave. Then the decision maker exhibits a stronger intertemporal risk aversion in period $t+1$, than he does in period t. For such a preference specification, $g_{t+1} \circ f_{t+1}^{-1}$ turns out to be the only nonlinear term in expression (10.2). Being a strictly increasing function, this inverse of $f_{t+1} \circ g_{t+1}^{-1}$ is convex. Thus, the decision maker prefers an early resolution of uncertainty. In general, however, the other two effects discussed before play an important role as well. They are 'sandwiched' between the terms $g_t \circ f_t^{-1}$ and $\left[g_{t+1} \circ f_{t+1}^{-1}\right]^{-1}$, which characterize the relative difference in intertemporal risk aversion between the two periods. The second effect, depending on how uncertainty aggregation is influenced by first period welfare is caught in general terms by $\theta_t \, g_t \circ u_t(x_t)$. The first effect, depending on how uncertainty aggregation is influenced by the devaluation of the second period, is expressed in general terms through the factor $\theta_t \theta_{t+1}^{-1}$.[12]

Turning around the driving forces for an early resolution, the three effects above work towards a preference for a late resolution of uncertainty. Straight forwardly, the reasons can be summarized as follows. First, the decision maker's discounted value of a lottery over undiscounted outcomes is higher than his value for a lottery over discounted outcomes. Second, the decision maker is more risk averse when evaluating a lottery with coinciding gains and losses at a higher welfare level, than he is evaluating it at a lower welfare level. Third, the decision maker is less intertemporally risk averse in the period of late uncertainty resolution than he is in the period of early uncertainty resolution. The next section discusses the question, whether a preference for early or late resolution of uncertainty on the grounds discussed above is desirable or even reasonable, in particular for a social decision maker.

10.2 Indifference to the Timing of Uncertainty Resolution & Reduction of Recursive Probabilites

In the preceding section I have elaborated that and how a decision maker, who applies the general recursive framework employed in chapters 8 and 9, can exhibit an intrinsic preference for the timing of uncertainty resolution. In particular, an intertemporally risk averse decision maker who subscribes to the generalized isoelastic setup, i.e. the model commonly applied to disentangle (atemporal) risk aversion from intertemporal substi-

[12] As theorem 5 has shown, the term $\theta_t \theta_{t+1}^{-1} \vartheta_t$ can be eliminated by an appropriate choice of the functions g_t. Therefore, the corresponding term in expression (10.2) does not introduce a driving force for a preference for the timing of uncertainty resolution, that is conceptually different from the three effects summarized above.

tutability, has to prefer an early resolution of uncertainty. In this section I question whether such an intrinsic preference for the timing of uncertainty resolution is reasonable. I suggest that such a preference might not be desirable in a principled approach to social decision making under uncertainty. In that case, the temporal information that is embedded in the recursive description of uncertainty by temporal lotteries is no longer needed for evaluation. I work out, how eliminating the timing information from recursive temporal lotteries yields a common description of uncertainty in terms of probability measures that are directly defined on consumption paths.

The most clear-cut motivation found in the literature for recursive utility with a non-trivial attitude towards the timing of uncertainty resolution is the comparison between lotteries A, B and C discussed on page 151 in the previous section. However, I have already pointed out that the next section proves that a strict preference for lottery A over lottery B can also be explained with a non-recursive representation under the assumption of indifference to the timing of uncertainty resolution. Therefore, to further analyze why a decision maker might prefer an early or late resolution of uncertainty, I concentrate on the comparison between lotteries A and C. Recall that in lottery A a coin is tossed at the beginning of every period. If head comes up, the decision maker receives the high payoff in the respective period and, if tail comes up, the decision maker faces the low payoff in the respective period. Lottery C has been defined largely analogous to lottery A, with the only difference that in lottery C coins for the outcome of every period are tossed already at the beginning of the first period. Therefore, lottery C corresponds to a lottery with an early resolution of uncertainty, while lottery A corresponds to a lottery with a late resolution of uncertainty. The only motivation given in the literature why one of these two lotteries might be preferred over the other is "based on the psychic costs and benefits of early resolution" (Duffie & Epstein 1992). Such a psychological explanation of a potential non-indifference between the two lotteries raises the suspicion that it might not be a desirable feature of a decision support model for a social decision maker. In the following, I argue that a social decision maker should be indifferent between lotteries A and C. To sharpen the point, let me introduce a lottery D. Lottery D is defined analogously to lottery C, only that the decision maker does not observe the coin tosses. Only at the beginning of every period, the decision maker is given the result of the coin toss that has decided over the outcome of the respective period. Then[13] lottery D is formally equivalent to lottery A. Now, a decision maker

[13]That is the case at least for the concepts of epistemic or subjective probabilities. For an objectivist's view on probability like in the case of the Popperian propensity, one might a well try to argue for detaching the uncertainty resolution from the person who receives the information. See chapter 5.2 for a brief overview of the different concepts of probability.

with a (strict) preference for early resolution of uncertainty is willing to reduce welfare, e.g. in period one, in order to exchange lottery D for lottery C. That is, he gives up welfare in order to obtain an information earlier, of which he knows that it is of no use. At least for a social decision maker, I do not think that such a behavior is desirable.

The lottery motivation for an intrinsic timing preference for uncertainty resolution discussed above, is the most formal found in the respective literature. In the following, I discuss two less formal motivations, which I regard the most elaborated and interesting ones that I have encountered. Chew & Epstein (1989, 108) give the example of a Ph.D. student who is about to spend a month of vacation in France. His comprehensive exams have already been graded. Now, he has to decide whether he wants to be informed about the result before or after his vacation (which he is committed to take). So far the example of Chew & Epstein (1989). Certainly, there are individuals who would rather be informed immediately, while others prefer not to possibly spoil their vacation with bad news. The latter formulation already suggests my perception of the example. I do not think that the reason why a student might prefer to receive the information before or after the vacation is in fact due to an intrinsic preference for early or late resolution of uncertainty, as it has been analyzed in the preceding section. I rather think that the outcomes in the two lotteries are actually different. Receiving the news of having failed the exam affects the welfare of the student beyond the future consequences of repeating the exam, or changing his career. I suppose, the student will be unhappy if is informed about having failed the exam. Moreover, the particular setting of the example in which he receives the information before or after a *vacation*, suggests that there is more to his 'timing preference' than the mere passage of time. I suggest that there is an interaction between the unhappiness due to the bad news and the ability to enjoy a vacation. For some people the bad news inhibits their ability to enjoy the vacation, while for others, the vacation can help to better digest the bad news. In consequence, the first type of people would prefer to receive the information after the vacation, while the second type would like to be informed right away. Therefore, I would favor a more explicit description of the welfare states at hand. If this is done, the preference for the timing of uncertainty resolution might no longer be intrinsic, but rather instrumental to avoid special types of information, which in some situations can affect welfare by itself and interact with other characteristics of well-being. The above example involves a personal information affecting a personal mood. For a principled approach to choice under uncertainty, in particular for a social decision maker, I am convinced that the timing should affect the decision only, if a welfare effect of the received information has clearly been elaborated and made explicit. However, than it is not intrinsic anymore and not due to the effects discussed in the previous section.

Another interesting example of a preference for an early or late resolution of uncertainty is given by Grant et al. (1998). The authors consider a situation where a person has the opportunity to be tested for an uncurable genetic disorder. Grant et al. (1998) cite a director of a genetic counseling program who states that "there are basically two types of people. There are 'want-to-knowers' and there are 'avoiders'. There are some people who, even in the absence of being able to alter outcomes, find information of this sort beneficial. The more they know, the more their anxiety level goes down. But there are others who cope by avoiding, who would rather stay hopeful and optimistic and not have the unanswered questions answered." My interpretation of the situation is similar to that of Chew & Epstein's (1989) example. The 'want-to-knowers' are described as people whose anxiety level goes down when they learn more about their potential disorders or diseases. Again it seems to me that the information more directly affects the welfare level than through any of the mechanisms discussed in the previous section, corresponding to an intrinsic preference for an early resolution of uncertainty as captured in the Kreps & Porteus (1978) model. On the other hand, the avoiders are described as hopeful and optimistic people who rather leave the question unanswered. Probably there are many reasons, why someone prefers to abstain from undergoing the genetic test. Let me explicitly work out the 'explanation' that an intrinsic preference for a late resolution of uncertainty in the generalized isoelastic decision model would give for the behavior of the 'avoiders'. The (avoiding) decision maker is aware that he might suffer at some point from an uncurable disease. The difference between being tested now or never would be the following. When he takes the test today, he includes all of his welfare experienced before he might fall ill into his lottery evaluation. Therefore, he evaluates the risk of suffering from an incurable disease at a higher aggregate welfare level. Moreover, the decision maker has to be intertemporally *risk seeking*. Then, as he exhibits a constant coefficient of relative intertemporal risk seekingness, he *prefers* to take the *risk at a lower welfare level*. In his recursive evaluation this is achieved if he *postpones* the resolution of the uncertainty as far as possible.[14] Thus, the decision maker prefers not to be tested at all. I doubt that such a mechanism underlying a preference for a late resolution of uncertainty reflects the motives of the 'avoiders'.

Finally, another argument, which is ubiquitous in the literature using or developing general recursive models relying on a non-trivial attitude with respect to the timing of uncertainty resolution, is that these models allow to disentangle between risk aversion and intertemporal substitutability. Weil (1990, 32) even states that "attitudes toward

[14]Then, the welfare experienced before falling ill is excluded from the lottery evaluation.

intertemporal substitution and risk aversion can be distinguished within the context of KP[15] preferences - precisely because these preferences do not impose indifference toward the timing of resolution of uncertainty over temporal lotteries." The next section, however, shows that it is possible to distinguish between intertemporal substitution and risk aversion in a non-recursive model satisfying indifference to the timing of uncertainty resolution. Let me conclude the discussion. While there are situations in which people exhibit preferences for an early or late resolution of uncertainty, I argue that these preferences are not well described by an intrinsic preference for the timing of uncertainty resolution as it has been analyzed in the previous section. Moreover, I believe that the examples found in the literature to motivate an intrinsic preference for the timing of uncertainty resolution are of interest for personal decision making rather than for a social decision maker. If comparable situations can arise in social decision making, I would advocate that the welfare effects of the information and its timing are stated explicitly, making the preference for an early or late resolution of uncertainty in these rather special situations instrumental. Finally, I believe that a principled approach to choice under uncertainty should not depend on whether certain outcomes before a risky period are considered part of the lottery or not, which is the essence of a corresponding preference in the sense of the last section. In consequence, I consider the following axiom as a desirable consistency requirement for a principled approach to choice under uncertainty.

A10 (indifference to the timing of risk resolution)
For all $t \in \{1, ..., T-1\}$, $x_t \in X$, $p_{t+1}, p'_{t+1} \in P_{t+1}$ and $\lambda \in [0,1]$ it holds

$$\lambda(x_t, p_{t+1}) + (1-\lambda)(x_t, p'_{t+1}) \quad \sim_t \quad (x_t, \lambda p_{t+1} + (1-\lambda)p'_{t+1}).$$

The axiom requires indifference to the timing of uncertainty resolution as it has been defined and discussed in the preceding section and above.

If axiom A10 is met, the information on temporal resolution of uncertainty contained in the recursive probability measures $p_{t, t\in\{1,...,T\}}$ is no longer needed for evaluative purposes.[16] In consequence, I can use 'common' probability measures that are defined directly on the space of future consumption paths X^t to describe the prevailing uncertainty. In the remaining part of this section, I show how these 'non-temporal' probability measures can be derived from its temporal counterparts. To this end, I inductively strip away the information on the timing of uncertainty resolution. Given a lottery

[15] Weil (1990) abbreviates Kreps & Porteus (1978) with KP.

[16] This intuition finds its formal validation in theorem 12 in the next section.

$p_t \in P_t, t \in \{1, ..., T\}$, this is done by deriving for all $\tau \in \{t, ..., T\}$ reduced probability distributions for the outcomes x_τ that are only conditioned on the previous outcome realizations $x_t, ..., x_{\tau-1}$. These conditional probabilities together render a probability measure on the set of consumption paths $p_t^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$. The latter contains information on how probable every outcome in a particular period is, but no information on the timing of uncertainty resolution. The following demonstration how to derive a 'common' or 'non-temporal' probability measure $p_t^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$ from a corresponding 'temporal' lottery $p_t \in P_t$ is not necessary to understand the representation and discussion in the subsequent sections. The less technically minded reader may skip this rather formal remainder of the section.

Pick any $p_t \in P_t$ with $t \in \{1, ..., T\}$ and define for inductive purposes $\mathbb{P}^{X_t, P_{t+1}} = p_t$. Moreover, denote with $\mathcal{L}(Y)$ the Borel $\sigma-$field of a metric space Y. By a probability distribution of $y \in Y$, I formally mean a probability measure defined on $\mathcal{L}(Y)$. Then, the marginal probability distribution of x_t is defined as

$$\mathbb{P}^{X_t}(A_t) = \mathbb{P}^{X_t, P_{t+1}}(A_t, P_t) \quad \forall A_t \in \mathcal{L}(X)\,.$$

To arrive at the distribution of x_{t+1}, I first define the conditional probability of p_{t+1} given x_t. The latter is formally defined through the probability kernel $\mathbb{P}^{P_{t+1}|X_t} : X \times \mathcal{L}(P_{t+1}) \to \mathbb{R}_+$ satisfying the requirement

$$\mathbb{P}^{X_t, P_{t+1}}(A_t, B_{t+1}) = \int_{A_t} \mathbb{P}^{P_{t+1}|X_t}(x_t, B_{t+1})\, d\mathbb{P}^{X_t}(x_t)$$

for all $A_{t+1} \in \mathcal{L}(X)$ and $B_{t+1} \in \mathcal{L}(P_{t+1})$. Then, for every $x_t \in X$ setting $\mathbb{P}^{P_{t+1}|x_t}(B_{t+1}) \equiv \mathbb{P}^{P_{t+1}|X_t = x_t}(B_{t+1}) \equiv \mathbb{P}^{P_{t+1}|X_t}(x_t, B_{t+1})$ defines a probability measure on P_{t+1}. For every $x_t \in X$ it gives a probability distribution over the temporal lotteries p_{t+1}, i.e. a probability distribution over probability distributions on $\tilde{X}_{t+1}$. This second order probability distribution $\mathbb{P}^{P_{t+1}|x_t}$ contains the information on the uncertainty resolving in period t.[17] In particular, if $p_t = (x_t, p_{t+1})$ is degenerate, implying that no uncertainty resolves in period t, then also $\mathbb{P}^{P_{t+1}|x_t}$ is degenerate: $\mathbb{P}^{P_{t+1}|x_t}(p_{t+1}) = 1$. To arrive at a reduced probability measure on $\tilde{X}_{t+1}$, I 'integrate out' the temporal information by summing over all 'positive weighted' measures p_{t+1}. This step yields for every $x_t \in X$ the reduced probability measure $\mathbb{P}^{X_{t+1}, P_{t+2}|x_t}$ on $\tilde{X}_{t+1}$ defined by

$$\mathbb{P}^{X_{t+1}, P_{t+2}|x_t}(A_{t+1}, B_{t+2}) = \int_{P_{t+1}} p_{t+1}(A_{t+1}, B_{t+2})\, d\mathbb{P}^{P_{t+1}|x_t}(p_{t+1})$$

for all $A_{t+1} \in \mathcal{L}(X)$ and $B_{t+2} \in \mathcal{L}(P_{t+2})$. Then, the probability distribution of x_{t+1}

[17] Precisely, only over the uncertainty resolving over period $t+1$ and later periods, given a particular outcome x_t in period t.

given x_t is obtained as the marginal of $\mathbb{P}^{X_{t+1},P_{t+2}|x_t}$ defined as

$$\mathbb{P}^{X_{t+1}|x_t}(A_{t+1}) = \mathbb{P}^{X_{t+1},P_{t+2}|x_t}(A_{t+1}, P_{t+2})$$

for all $A_{t+1} \in \mathcal{L}(X)$.

The same procedure to obtain conditional probabilities $\mathbb{P}^{X_\tau|x_{\tau-1},...,x_t}$ can be carried out *inductively* for all $\tau \in \{t+1, ..., T-1\}$. The induction step works as follows. Let $\mathbb{P}^{X_\tau,P_{\tau+1}|x_{\tau-1},...,x_t}$ and its marginal $\mathbb{P}^{X_\tau|x_{\tau-1},...,x_t}$ be given for period τ. Then, define for any given sequence $x_t, ..., x_{\tau-1}$ the conditional probability of $p_{\tau+1}$ given x_τ through the probability kernel $\mathbb{P}^{P_{\tau+1}|x_{\tau-1},...,x_t} : X \times \mathcal{L}(P_{\tau+1}) \to \mathbb{R}_+$ satisfying the requirement

$$\mathbb{P}^{X_\tau,P_{\tau+1}|x_{\tau-1},...,x_t}(A_\tau, B_{\tau+1}) = \int_{A_\tau} \mathbb{P}^{P_{\tau+1}|X_\tau,x_{\tau-1},...,x_t}(x_\tau, B_{\tau+1})\, d\mathbb{P}^{X_\tau|x_{\tau-1},...,x_t}(x_\tau)$$

for all $A_\tau \in \mathcal{L}(X)$ and $B_{\tau+1} \in \mathcal{L}(P_{\tau+1})$. Then for every sequence $x_t, ..., x_\tau$, setting $\mathbb{P}^{P_{\tau+1}|x_\tau,...,x_t}(B_{\tau+1}) \equiv \mathbb{P}^{P_{\tau+1}|X_\tau=x_\tau,...,x_t}(B_{\tau+1}) \equiv \mathbb{P}^{P_{\tau+1}|X_\tau,x_{\tau-1},...,x_t}(x_\tau, B_{\tau+1})$ defines a probability measure on $P_{\tau+1}$. Again, this measure is a probability distribution over the probability distributions $p_{\tau+1}$ on $\tilde{X}_{\tau+1}$, containing information on the uncertainty resolving in period τ. To arrive at the reduced probability measure on $\tilde{X}_{\tau+1}$, I 'integrate out' the temporal information by summing over all weighted measures $p_{\tau+1}$. This step yields for every sequence $x_t, ..., x_\tau$ the reduced probability measure $\mathbb{P}^{X_{\tau+1},P_{\tau+2}|x_\tau,...,x_t}$ defined by

$$\mathbb{P}^{X_{\tau+1},P_{\tau+2}|x_\tau,...,x_t}(A_{\tau+1}, B_{\tau+2}) = \int_{P_{\tau+1}} p_{\tau+1}(A_{\tau+1}, B_{\tau+2})\, d\mathbb{P}^{P_{\tau+1}|x_\tau,...,x_t}(p_{\tau+1})$$

for all $A_{\tau+1} \in \mathcal{L}(X), B_{\tau+2} \in \mathcal{L}(P_{\tau+2})$. Finishing the induction step, I find the probability of $x_{\tau+1}$ given $x_t, ..., x_\tau$ as the marginal

$$\mathbb{P}^{X_{\tau+1}|x_\tau,...,x_t}(A_\tau) = \mathbb{P}^{X_{\tau+1},P_{\tau+2}|x_\tau,...,x_t}(A_{\tau+1}, P_{\tau+2})$$

for all $A_{\tau+1} \in \mathcal{L}(X)$.

That way, I arrive for $\tau = T-1$, and for any sequence $x_t, ..., x_{T-2}$, at the measure $\mathbb{P}^{X_{T-1},P_T|x_{T-2},...,x_t}$ on $\tilde{X}_{t-1}$, and its marginal $\mathbb{P}^{X_{T-1}|x_{T-2},...,x_t}$. In the final step, I obtain as before $\mathbb{P}^{P_T|x_{T-1},...,x_t}$ as the conditional probability of p_T given $x_t, ..., x_{T-1}$. However, this time, the conditional probability of x_T given $x_t, ..., x_{T-1}$ is obtained directly from 'integration out' p_T:

$$\mathbb{P}^{X_T|x_{T-1},...,x_t}(A_T) = \int_{P_T} p_T(A_T)\, d\mathbb{P}^{P_T|x_{T-1},...,x_t}(p_T)$$

for all $A_T \in \mathcal{L}(X)$. Having derived the conditional probability measures $\mathbb{P}^{X_\tau|x_{\tau-1},...,x_t}$ for all periods $\tau \in \{t, ..., T\}$, and all sequences $x_t, ..., x_{\tau-1}$, I obtain the desired probability measure on the set of certain consumption paths $p^{\mathbf{X}_t}$, as the composition of the derived conditional probabilities. Formally, define $p_t^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$ by the requirement

$$\begin{aligned} p_t^{\mathsf{X}}(A_t, ..., A_T) &= \prod_{\tau=t}^{T} \int_{A_\tau} d\mathbb{P}^{X_\tau | x_{\tau-1},...,x_t}(x_\tau) \\ &= \int_{A_t \times ... \times A_T} d\mathbb{P}^{X_T | x_{T-1},...,x_t}(x_T) \, ... \, d\mathbb{P}^{X_{t+1}|x_t}(x_{t+1}) \; d\mathbb{P}^{X_t}(x_t) \end{aligned}$$

for all $A_t, ..., A_T \in \mathcal{L}(X)$. By construction, the measure p_t^{X} comprises the non-temporal information of the lottery p_t. For a representation of preferences respecting axiom A10, only this information is needed to evaluate the uncertain future. The latter assertion is verified as part of theorem 12, which is stated in the next section.

10.3 Intertemporal Risk Aversion and Non-Recursive Uncertainty

The section works out preference representations for a decision maker who is indifferent to the timing of uncertainty resolution in the sense discussed in the previous section Adding axiom A10 to the assumptions of chapter 8 yields the following theorem.

Theorem 12: Let there be given a sequence of preference relations $(\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ and a sequence of Bernoulli utility functions $(u_t)_{t\in\{1,...,T\}}$ with $u_t \in B_{\succeq_t}$. The sequence of preference relations $(\succeq_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting)

ii) A4' for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) A10 (timing indifference)

if and only if for all $t \in \{1, ..., T\}$ there exist strictly increasing continuous functions $g_t : U_t \to \mathbb{R}$, as well as a function $h \in \left\{ \exp, \text{id}, \frac{1}{\exp} \right\}$, such that with defining

v) the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by

$$\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^{T} g_\tau \circ u_\tau(\mathsf{x}_\tau^t) \tag{10.6}$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p_t' \;\Leftrightarrow\; \mathcal{M}^h(p_t^{\mathsf{X}}, \tilde{u}_t) \geq \mathcal{M}^h(p_t'^{\mathsf{X}}, \tilde{u}_t) \quad \forall\, p_t, p_t' \in P_t, \tag{10.7}$$

Moreover, in a representation employing $h \in \{\exp, \frac{1}{\exp}\}$, two sequences $(g_t)_{t\in\{1,...,T\}}$ and $(g'_t)_{t\in\{1,...,T\}}$ both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, there exists $b_t \in \mathbb{R}$ for every $t \in \{1, ..., T\}$, such that $g_t = g'_t + b_t$. In a representation employing $h = \text{id}$, two sequences $(g_t)_{t\in\{1,...,T\}}$ and $(g'_t)_{t\in\{1,...,T\}}$ both represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the above sense, if and only if, there exists $a \in \mathbb{R}_{++}$, as well as $b_t \in \mathbb{R}$ for every $t \in \{1, ..., T\}$, such that $g_t = ag'_t + b_t$.

As anticipated in the previous section, the assumption of timing indifference implies that the evaluation in equation (10.7) only employs aggregate utility $\tilde{u}_t$ and probability measures $p_t^{\mathsf{X}}, p_t'^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$ that are defined non-recursively over consumption paths. The measures $p_t^{\mathsf{X}}, p_t'^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$ are derived from their recursive counterparts $p_t, p'_t \in P_t$ the way explained in section 10.2. This relation, however, is only needed to axiomatize the representation in the general setup. For an application of the theorem, it is sufficient to describe the uncertain future directly with the measure $p_t^{\mathsf{X}} \in \Delta(\mathsf{X}^t)$. In view of the axioms, note that in the above setting a result by Chew & Epstein (1989, 110) allows to replace the independence axiom A3 by a collection of weaker assumptions.

To identify intertemporal risk aversion, I have to fix the measure scale for welfare in the representation. Different possibilities for doing so in the non-stationary setting have been explored in lemma 5. A full fixing of the measure scale corresponds to cases c) and d) of the lemma and brings about the existence of $\xi \in \mathbb{R}$ such that with the same definition of $\tilde{u}_t$ as in equation (10.6), the representing equation (10.7) becomes[18]

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{\exp^\xi}(p_t^{\mathsf{X}}, \tilde{u}_t) \geq \mathcal{M}^{\exp^\xi}(p_t'^{\mathsf{X}}, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t\,.$$

For such a u^+-gauge, the measures of intertemporal risk aversion are defined uniquely as $\text{AIRA}_t = -\frac{\xi}{\theta_t}$ and $\text{RIRA}_t = -\frac{\xi}{\theta_t}\,\text{id}$. In the g^+-gauge the corresponding result is stated in the following corollary.

Corollary 11 **($g = \textbf{id}^+-$gauge, non-stationary)** : Choose numbers $\underline{w}_t \in \mathbb{R}$ for all $t \in \{1, ..., T\}$ (minimum welfare levels), as well as $t^* \in \{1, ..., T\}$ and a number $\overline{w}_{t^*} \in \mathbb{R}$ satisfying $\overline{w}_{t^*} > \underline{w}_{t^*}$ (maximum welfare level in period t^*).
Then, a sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

$i) - iv)$ of theorem 12,

if and only if, there exist continuous functions $u_t : X \to U_t$ with $\underline{U}_t = \underline{w}_t$ for all $t \in \{1, ..., T\}$ and $\overline{U}_{t^*} = \overline{w}_{t^*}$, as wells as $\xi \in \mathbb{R}$, such that with defining

[18]See equation (C.46) in the proof of theorem 12.

v) the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by

$$\tilde{u}_t(\mathrm{x}^t) = \sum_{\tau=t}^{T} u_\tau(\mathrm{x}^t_\tau)$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{\exp^\xi}(p_t^{\mathbf{x}}, \tilde{u}_t) \geq \mathcal{M}^{\exp^\xi}(p_t'^{\mathbf{x}}, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t\,. \tag{10.8}$$

Moreover, the functions $u_{t,t\in\{1,...,T\}}$ are determined uniquely, as are the measures of intertemporal risk aversion $\text{AIRA}_t = -\frac{\xi}{\theta_t}$ and $\text{RIRA}_t = -\frac{\xi}{\theta_t}\,\text{id}$, where θ_t is the time dependent normalization constant defined in theorem 4, here $\theta_t = \frac{\Delta U_t}{\sum_{\tau=t}^{T}\Delta U_t}$.

Observe that uncertainty in the representing equation (10.8) is evaluated with *intertemporal* risk attitude. Let me relate the result to the discussion in section 10.1. There, I have pointed out that in the case of an early resolution of uncertainty, certain outcomes before the risky period are considered as part of the lottery. Thus, if the preceding period yields a non-zero welfare, the uncertainty is evaluated at a different welfare level than in the case of a late resolution of uncertainty. If (intertemporal) risk attitude depends on the welfare level, this difference causes a preference for either of the two lotteries. Only if (absolute) intertemporal risk attitude is independent of the welfare level, also the uncertainty evaluation is independent of whether certain outcomes in preceding periods are conceived as part of the lottery or not. Only then, indifference to the timing of uncertainty prevails. Therefore AIRA_t is constant. Moreover, chapter 10.1 has elaborated that a difference in the attitude of intertemporal risk aversion between different periods can imply a propensity to have uncertainty resolved in the period with the lower intertemporal risk aversion. Therefore, the coefficients of intertemporal risk aversion in the above representation only depend on the relative weight given to a particular period as opposed to the remaining future, as characterized by the normalization constant θ_t, but not otherwise on time. This difference to the representations derived under the assumption of risk stationarity in chapter 9 will be discussed in the next section.

The following representation adds the assumption of certainty stationarity formulated as axiom A7 in chapter 9.1. I state the representation directly in the $g-$gauge where it implies the standard discount utility evaluation on certain consumption paths, and for a fixed measure scale of welfare.

Theorem 13 **($g = \mathbf{id}^+-$gauge, certainty stationary) :** Choose a nondegenerate closed interval $W^* \subset \mathbb{R}_{++}$.

A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting) *ii*) A4' for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) A7 & A10 (certainty stationarity & timing indifference)

if and only if, there exists a continuous and surjective function $u : X \to W^*$, a discount factor $\beta \in \mathrm{I\!R}_{++}$ and $\xi \in \mathrm{I\!R}$, such that with defining

v) the functions $\tilde{u}_t : \tilde{X}_t \to \mathrm{I\!R}$ for $t \in \{1, ..., T\}$ by

$$\tilde{u}_t(\mathrm{x}^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} u(\mathrm{x}_\tau^t)$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{\exp^\xi}(p_t^{\mathrm{X}}, \tilde{u}_t) \geq \mathcal{M}^{\exp^\xi}(p_t'^{\mathrm{X}}, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t\,.$$

Moreover, the function u is determined uniquely, as are the measures of intertemporal risk aversion $\mathrm{AIRA}_t = -\frac{\xi}{1-\beta_t}$ and $\mathrm{RIRA}_t = -\frac{\xi}{1-\beta_t}\,\mathrm{id}$.

In all periods outcomes are evaluated with a common certainty additive Bernoulli utility function u, which describes welfare in the sense of chapter 7.3. Overall evaluation of a particular consumption path is performed by taking the discounted sum of per period welfare. To evaluate an uncertain future, the decision maker weights the aggregate welfare of the possible consumption paths with their respective probabilities, and applies the uncertainty aggregation rule $\mathcal{M}^{\exp^\xi}$, which is parametrized (up to a normalization factor) by the coefficient of absolute intertemporal risk seeking, i.e. the negative of absolute intertemporal risk aversion. For the limit of an infinite time horizon, the normalization constant that depicts the relative weight of an individual period as opposed to the remaining future $1 - \beta_t$ becomes constant over time. In consequence, so does the coefficient of intertemporal risk aversion $\lim_{T\to\infty} \mathrm{AIRA}_t = -\frac{\xi}{1-\beta}$. For a finite time horizon, as the end of the planning horizon is approached, the decreasing length of the welfare paths under consideration goes along with a coefficient of absolute intertemporal risk aversion AIRA_t that decreases over time to $-\xi$ for the last period. Note that, in accordance with the convention underlying lemma 7, the measure scale for welfare has been fixed to W^* in period 1, implying ranges $\beta^{t-1}W^*$ for welfare measurement in later periods.

In particular, theorem 13 shows that it is possible to disentangle atemporal risk aversion from intertemporal substitutability without assuming an intrinsic preference for early or late resolution of uncertainty. In addition, such a model is compatible with the widespread discount utility model for the evaluation of individual consumption paths.

The possibility to disentangle these two characteristics of preference, defined in a one commodity setting, follows immediately from the fact that the coefficients of intertemporal risk aversion characterize a difference between intertemporal and atemporal uncertainty aggregation. Before I state the Epstein Zin gauge, in which the coefficients of relative (atemporal) risk aversion and intertemporal substitutability have been discussed in chapter 7.1, I give some alternative formulations of general multiperiod representation.

Representing the same preferences described in theorem 13 in the Kreps Porteus gauge yields the alternative form

Corollary 12 **($f = \mathrm{id}^{+}$−gauge, certainty stationary)** : Choose a nondegenerate closed interval $U^* \subset \mathbb{R}_{++}$. A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

$i) - iv)$ of theorem 13,

if and only if, there exists a continuous and surjective function $u : X \to U^*$, a discount factor $\beta \in \mathbb{R}_{++}$ and $\xi \in \mathbb{R}$, such that with defining

$v)$ the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$

- for $\xi > 0$ by $\tilde{u}_t(\mathrm{x}^t) = \prod_{\tau=t}^{T} u(\mathrm{x}_\tau^t)^{\xi\beta^{\tau-1}}$ and
- for $\xi = 0$ by $\tilde{u}_t(\mathrm{x}^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} \ln u(\mathrm{x}_\tau^t)$ and
- for $\xi < 0$ by $\tilde{u}_t(\mathrm{x}^t) = -\prod_{\tau=t}^{T} u(\mathrm{x}_\tau^t)^{\xi\beta^{\tau-1}}$

the following equation holds for all $t \in \{1, ..., T\}$

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathrm{E}_{p_t^{\mathrm{x}}}\, \tilde{u}_t(\mathrm{x}^t) \geq \mathrm{E}_{p_t'^{\mathrm{x}}}\, \tilde{u}_t(\mathrm{x}^t) \quad \forall\, p_t, p'_t \in P_t,$$

Moreover, the function u is determined uniquely, as are the measures of intertemporal risk aversion $\mathrm{AIRA}_t = -\frac{\xi}{1-\beta_t}$ and $\mathrm{RIRA}_t = -\frac{\xi}{1-\beta_t}\,\mathrm{id}$.

Here, uncertainty is evaluated by the expected value operator. As observed in the earlier models, this linearization of uncertainty aggregation comes at the price of introducing a nonlinear aggregation of per period utility over time.[19] Making the latter purely

[19] Also in corollary 12, the measure scale for welfare is fixed for period 1. However, note that due to the multiplicative form of intertemporal aggregation, the range of welfare u^{welf} in the certainty additive sense of chapter 7.3 is fixed to the range $W^* = \ln U^*$ rather than to the range U^*. This is also the reason, why a logarithm appears in the representation for $\xi = 0$. Only with this definition, the range of welfare is fixed independently of the parameter ξ. However, eliminating the logarithm would not change the measures of intertemporal risk aversion, as they are zero in the case $\xi = 0$ and, thus, independent of the particular measure scale adopted for welfare.

multiplicative over time brings about an isoelastic uncertainty aggregation rule, as stated in the following variation of corollary 12.

Corollary 13 (isoelastic uncertainty evaluation, certainty stationary) :
Choose a nondegenerate closed interval $U^* \subset \mathbb{R}_{++}$. A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

$i) - iv)$ of theorem 12,

if and only if, there exists a continuous and surjective function $u : X \to U^*$, a discount factor $\beta \in \mathbb{R}_{++}$ and $\xi \in \mathbb{R}$, such that with defining

$v)$ the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by

$$\tilde{u}_t(\mathrm{x}^t) = \prod_{\tau=t}^{T} u(\mathrm{x}^t_\tau)^{\beta^{\tau-1}}$$

the following equation holds for all $t \in \{1, ..., T\}$

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^\xi(p^{\mathrm{X}}_t, \tilde{u}_t) \geq \mathcal{M}^\xi(p'^{\mathrm{X}}_t, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t,$$

Moreover, the function u is determined uniquely, as are the measures of intertemporal risk aversion $\text{AIRA}_t = -\frac{\xi}{1-\beta_t}$ and $\text{RIRA}_t = -\frac{\xi}{1-\beta_t}\,\text{id}$.

Note that for a one commodity setting, the assumption $u = \text{id}$ in the above representation corresponds to the assumption of logarithmic welfare in the certainty additive model. I want to close this collection of alternative representations for preferences satisfying certainty stationarity and indifference to the timing uncertainty resolution by moving more generally to the one commodity scenario and stating the Epstein-Zin gauge. Here the (cardinal) consumption level is assumed to be a subset of $\mathbb{R}$, and Bernoulli utility is assumed to be strictly increasing in $x \in X \subset \mathbb{R}$.

Corollary 14 (one commodity $u = \text{id}^+$–gauge, certainty stationary) :
Choose a nondegenerate closed interval $W^* \subset \mathbb{R}$. A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

$i) - iv)$ of theorem 13,

if and only if, there exists a continuous and surjective function $g : \mathbb{R} \to W^*$, a discount factor $\beta \in \mathbb{R}_{++}$ and $\xi \in \mathbb{R}$, such that with defining

$v)$ the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by

$$\tilde{u}_t(\mathrm{x}^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} g(\mathrm{x}^t_\tau) \tag{10.9}$$

the following equation holds for all $t \in \{1, ..., T\}$

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^\xi(p_t^{\mathsf{X}}, \tilde{u}_t) \geq \mathcal{M}^\xi(p_t'^{\mathsf{X}}, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t,$$

Moreover, the function u id determined uniquely, as are the measures of intertemporal risk aversion $\text{AIRA}_t = -\frac{\xi}{1-\beta_t}$ and $\text{RIRA}_t = -\frac{\xi}{1-\beta_t}\,\text{id}$.

Note that for the non-stationary representation, equation (10.9) would change to the form[20]

$$\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^{T} g_t(\mathsf{x}_\tau^t)\,.$$

It has been worked out in chapter 7.1 that in the Epstein-Zin gauge g characterizes intertemporal substitutability, while f, here f_t, characterizes (atemporal) risk attitude. As f_t is not observed directly in the representation it has to be inferred from the relation characterizing intertemporal risk aversion (for $\xi \neq 0$) as[21]

$$\begin{aligned}
f_t g_t^{-1}(z) &= a_t \exp\left(\frac{\xi}{1-\beta_t}\, z\right) + b_t \\
\Rightarrow \quad f_t(z) &= a_t \exp\left(\frac{\xi}{1-\beta_t}\, g_t\right) + b_t = a_t \exp\left(\frac{\xi}{1-\beta_t}\, \beta^t g\right) + b_t
\end{aligned}$$

with $a_t, b_t \in \mathbb{R}$ and $a_t\,\xi > 0$. For the non-stationary setting analogously the relation $f_t(z) = a_t \exp(\frac{\xi}{\theta_t}\, g_t) + b_t$ holds.[22] Noting that in the $u = \text{id}$ gauge g and f_t are defined directly on $X \subset \mathbb{R}$, the coefficient of relative (atemporal) risk aversion calculates to

$$\begin{aligned}
\text{RRA}(x) &= -\frac{f''(x)}{f'(x)}\, x = -\frac{\frac{d^2}{dx^2} k_t \exp\left(\frac{\xi}{\theta_t} g_t(x)\right) + d_t}{\frac{d}{dx} k_t \exp\left(\frac{\xi}{\theta_t} g_t(x)\right) + d_t}\, x \\
&= -\frac{\left(\frac{\xi}{\theta_t} g'_t(x)\right)^2 \exp\left(\frac{\xi}{\theta_t} g_t(x)\right) + \frac{\xi}{\theta_t} g''_t(x) \exp\left(\frac{\xi}{\theta_t} g_t(x)\right)}{\frac{\xi}{\theta_t} g'_t(x) \exp\left(\frac{\xi}{\theta_t} g_t(x)\right)}\, x \\
&= -\left[\frac{\xi}{\theta_t} g'_t(x) + \frac{g''_t(x)}{g'_t(x)}\right] x \qquad (10.10) \\
&= -\left[\xi \frac{\beta^t}{1-\beta_t} g'(x) + \frac{g''(x)}{g'(x)}\right] x\,. \qquad (10.11)
\end{aligned}$$

Expression (10.10) holds for general non-stationary representations while for the cer-

[20]See equation (C.52) in the proof of corollary 14.

[21]This relation implying the stated coefficients of intertemporal risk aversion is derived in the proof of theorem 13 as equation (C.48).

[22]See equation (C.43) in the proof of theorem 12.

tainty stationary case given in corollary 14, where $\theta_t = 1 - \beta_t$ and $g_t = \beta^t g$, expression (10.11) applies. For example, the widespread assumption of logarithmic welfare, corresponding to $g = \ln$ and an intertemporal elasticity of substitution of unity, yields the particularly simple coefficient

$$\text{RRA}(x) = 1 - \xi \frac{\beta^t}{1-\beta_t} .$$

A form that corresponds to the isoelastic (atemporal) uncertainty aggregation rule $\mathcal{M}^\alpha$ with the time dependent coefficient $\alpha_t = \xi \frac{\beta^t}{1-\beta_t}$.[23] For an interpretation of the time dependence, I refer again to the subsequent section. Before, I want to keep a promise given in the preceding section.

In chapter 10.1 I have introduced two lotteries which are used in the literature to motivate a non-trivial (intrinsic) preference to the timing of uncertainty resolution. These lotteries have been defined as follows. In lottery A a coin is tossed at the beginning of *every* period. If head comes up, the decision maker receives the high payoff in the respective period and, if tail comes up, the decision maker faces the low payoff in the respective period. In contrast, in lottery B a coin is tossed *once* at the beginning of the first period. If head comes up, the agent receives the high payoff in *all* periods, if tail comes up the agents receives the low payoff in *all* periods. The intuition appealed to in the literature is that people would usually prefer lottery A over lottery B. In section 10.2 I have claimed that a preference for a late resolution of uncertainty is by no means necessary in order to explain the ranking. In the following I show that lottery A is preferred over lottery B also by a timing indifferent decision maker, if (and only if) he is intertemporally risk averse. It is easily recognized that the essence of such a preference is captured already in a two period model. Denote the high lottery outcome by $\overline{x}$ and the low lottery outcome by $\underline{x}$. Then, lottery A writes as $\frac{1}{2}(\overline{x}\,, \frac{1}{2}\overline{x} + \frac{1}{2}\underline{x}) + \frac{1}{2}(\underline{x}\,, \frac{1}{2}\overline{x} + \frac{1}{2}\underline{x})$. Lottery B is formally represented by $\frac{1}{2}(\overline{x}\ , \overline{x}) + \frac{1}{2}(\underline{x}\ , \underline{x})$. First, observe that in the intertemporally additive expected utility model indifference between the two lotteries prevails. Here, lottery A is evaluated by the expression

$$\tfrac{1}{4}\big(u(\overline{x}) + \beta u(\overline{x})\big) \;+\; \tfrac{1}{4}\big(u(\overline{x}) + \beta u(\underline{x})\big) \;+\; \tfrac{1}{4}\big(u(\underline{x}) + \beta u(\overline{x})\big) \;+\; \tfrac{1}{4}\big(u(\underline{x}) + \beta u(\underline{x})\big)$$

which is equivalent to

$$\tfrac{1}{2}\big(u(\overline{x}) + \beta u(\overline{x})\big) \;+\; \tfrac{1}{2}\big(u(\underline{x}) + \beta u(\underline{x})\big) ,$$

representing an intertemporally additive expected utility representation of lottery B. In contrast, for an intertemporally risk averse decision maker ($\xi < 0$) applying the

[23] Note that this case corresponds to the mentioned representation obtained from corollary 13 by assuming $u = \text{id}$.

evaluation characterized in theorem 13, the following relation holds:

$$\begin{aligned}
&\tfrac{1}{2}(\bar{x}\,,\tfrac{1}{2}\bar{x}+\tfrac{1}{2}\underline{x}) \;+\; \tfrac{1}{2}(\underline{x}\,,\tfrac{1}{2}\bar{x}+\tfrac{1}{2}\underline{x}) \;\succ\; \tfrac{1}{2}(\bar{x}\,,\bar{x}) \;+\; \tfrac{1}{2}(\underline{x}\,,\underline{x})\\
\Leftrightarrow\;& \tfrac{1}{\xi}\ln\Big[\tfrac{1}{4}\exp\big[\xi\big(u(\bar{x})+\beta u(\bar{x})\big)\big] \;+\; \tfrac{1}{4}\exp\big[\xi\big(u(\bar{x})+\beta u(\underline{x})\big)\big]\\
&\quad+\tfrac{1}{4}\exp\big[\xi\big(u(\underline{x})+\beta u(\bar{x})\big)\big] \;+\; \tfrac{1}{4}\exp\big[\xi\big(u(\underline{x})+\beta u(\underline{x})\big)\big]\Big]\\
&> \tfrac{1}{\xi}\ln\Big[\tfrac{1}{2}\exp\big[\xi\big(u(\bar{x})+\beta u(\bar{x})\big)\big] \;+\; \tfrac{1}{2}\exp\big[\xi\big(u(\underline{x})+\beta u(\underline{x})\big)\big]\Big]\\
\Leftrightarrow\;& \tfrac{1}{4}\exp\big[\xi\big(u(\bar{x})+\beta u(\underline{x})\big)\big] \;+\; \tfrac{1}{4}\exp\big[\xi\big(u(\underline{x})+\beta u(\bar{x})\big)\big]\\
&< \tfrac{1}{4}\exp\big[\xi\big(u(\bar{x})+\beta u(\bar{x})\big)\big] \;+\; \tfrac{1}{4}\exp\big[\xi\big(u(\underline{x})+\beta u(\underline{x})\big)\big]\\
\Leftrightarrow\;& \exp\big(\xi u(\underline{x})\big)\;\big[\exp\big(\xi\beta u(\bar{x})\big)-\exp\big(\xi\beta u(\underline{x})\big)\big]\\
&< \exp\big(\xi u(\bar{x})\big)\;\big[\exp\big(\xi\beta u(\bar{x})\big)-\exp\big(\xi\beta u(\underline{x})\big)\big]\\
\Leftrightarrow\;& 0<\big[\exp\big(\xi u(\bar{x})\big)-\exp\big(\xi u(\underline{x})\big)\big]\;\big[\exp\big(\xi\beta u(\bar{x})\big)-\exp\big(\xi\beta u(\underline{x})\big)\big]
\end{aligned}$$

If the decision maker is not indifferent between outcomes $\bar{x}$ and $\underline{x}$, i.e. if $u(\bar{x}) \neq u(\underline{x})$, the relation in the last line is always satisfied, as either both terms in the product are strictly positive, or both terms are strictly negative. In consequence, an intertemporally risk averse decision maker always prefers lottery A over lottery B. For an intertemporally risk seeking decision maker ($\xi > 0$), the inequality sign does not flip around in the step from the second to the third equivalence. Therefore, the opposite preference holds true and such a decision maker always prefers lottery B over lottery A. Only an intertemporally risk neutral decision maker, characterized by the intertemporally additive expected utility model, is indifferent between the two lotteries described above.

10.4 Implications for Discounting

In the preceding section I have shown that the requirement of indifference to the timing of uncertainty resolution is compatible with strict intertemporal risk aversion and a discount utility evaluation of certain consumption paths. This section analyzes the consequences of merging the assumption of indifference to the timing of uncertainty resolution with that of *risk stationarity* formulated in chapter 9.3 (risk stationarity II).

In theorem 13, I have described how certainty stationarity determines the time development of intertemporal risk aversion for a decision maker who has no intrinsic preference for early or late resolution of uncertainty. The coefficient of absolute intertemporal risk aversion was seen to be constant in welfare and to adapt to the length of the planning horizon lying ahead of the decision maker. It was calculated to $\text{AIRA}_t = -\frac{\xi}{1-\beta_t}$. Similarly, the assumption of risk stationarity formulated in axiom A9 gives rise to a coefficient of absolute intertemporal risk aversion that is constant in welfare. Moreover, the respective representation stated in corollary 8 exhibits the same adaption of the coefficients of intertemporal risk aversion to the length of the remaining planning horizon through the factor $\frac{1}{1-\beta_t}$. However, in contrast to the representation of the preceding section, for a decision maker who complies with risk stationarity, the coefficient of absolute intertemporal risk aversion also depends on the discount factor β^t. As worked out in chapter 9.3, under the assumption of axiom A9 only the functions $f_t \circ g^{-1}$ stay constant over time (up to the normalization by $\frac{1}{1-\beta_t}$). These functions, to which I have referred as the stationary characterization of intertemporal risk attitude, measure intertemporal risk aversion with respect to a 'current value measure scale for welfare'. In contrast, the coefficient AIRA_t expresses intertemporal risk aversion with respect to the 'present value measure scale for welfare'. That is, if the measure scale for period 1 is fixed to $\text{range}(u_1^{\text{welf}}) = W^*$, then the measure scale of welfare in period t shrinks down to the $\text{range}(u_t^{\text{welf}}) = \beta^{t-1}W^*$. But then, as the range of welfare measurement (in present value) becomes smaller and smaller over time due to discounting, the coefficient of intertemporal risk aversion has to increase in order to keep up a stationary aversion to risk. However, this is not allowed by axiom A10. If indifference to the timing of uncertainty resolution should prevail, the latter requires intertemporal risk aversion to be constant over time (up to the normalization by $\frac{1}{1-\beta_t}$). Otherwise, a decision maker would be willing to give up welfare in order to have uncertainty resolved in the period with the lowest intertemporal risk aversion, even if the information obtained is known to be of no use.

In consequence, risk stationary devaluation of the future, which implies by axiom A9 a decreasing coefficient of absolute intertemporal risk aversion, is not compatible with the demand of axiom A10, i.e. the lack of an intrinsic preference of uncertainty resolution. Precisely, there is only one situation where such a devaluation of the future is compatible with both axioms. For a decision maker who is intertemporally risk neutral, the assumption of risk stationarity has no more bite than the assumption of certainty stationarity. Here, the coefficient $\text{AIRA}_t = 0$ for all $t \in \{1, ..., T\}$ is constant over time and, thus, the intertemporally additive expected utility model trivially satisfies the requirements implied by both axioms. However, for a nontrivial model of intertemporally risk averse decision making, the following result obtains.

Theorem 14: A sequence of binary relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting) *ii)* A4' for $\succeq_1|_{X^T}$ (certainty additivity)

iii) A5' (time consistency)

iv) $A6^{s}_{st}$ (strict intertemporal risk aversion)

v) A9 (risk stationarity II)

vi) A10 (timing indifference)

if and only if, there exists a representation in the sense of theorem 13 and its corollaries with $\xi < 0$ and $\beta = 1$.

In words, a decision maker who accepts the above axioms does not discount the future due to an intrinsic timing preference. However, he does devaluate uncertain welfare[24]. In consequence, if uncertainty increases over time, future welfare gains less weight than current welfare. The remainder of this paragraph renders the latter intuition precise. For this purpose let $p_1^{\mathsf{X}} \in \Delta(\mathsf{X}^1)$ be a product measure $p_1^{\mathsf{X}} = \mathbb{P}_1^{x_1} \otimes ... \otimes \mathbb{P}_T^{x_T}$, so that the outcomes in different periods are independently distributed. Assume that expected welfare is the same in all periods, i.e. $E_{\mathbb{P}_t^{x_t}}\, u^{\text{welf}}(x_t) = \mathfrak{u}^* \in U = W^* \, \forall\, t \in \{1, ..., T\}$. To define what it means that uncertainty over welfare is increasing over time, I employ Rothschild & Stiglitz's (1970) definition of increasing risk. The authors define a random variable to be riskier than another, if the corresponding probability distribution has more weight on the tails.[25] In particular, this condition is satisfied for a mean preserving spread. Now, consider the probability distributions $\mathbb{P}_t^u$ over U that are induced by the measures $\mathbb{P}_t^{x_t}$ through the certainty additive Bernoulli utility function u^{welf}. Then, uncertainty of welfare increases over time, if $\mathbb{P}_t^u$ has more weight in the tails than $\mathbb{P}_{t'}^u$ for all $t, t' \in \{1, ..., T\}$ satisfying $t > t'$.[26] For such an uncertainty specification it follows from theorem 2 in Rothschild & Stiglitz (1970, 237) that the certainty equivalent of welfare in period t is lower than the certainty equivalent of welfare in period t'. As the expected welfare is the same in both periods, the difference in weights exhibits some resemblance to discounting. Note, in particular, that the intertemporally additive expected utility model does not allow for intertemporal risk aversion and, thus, not for risk aversion on welfare and the reasoning I carried out above. Therefore, the only

[24] I still adhere to the notion of welfare discussed in chapter 7.3 as certainty additive Bernoulli utility.

[25] An equivalent characterization is that the riskier random variable can be obtained from the less risky random variable by adding some noise. For a formal definition compare footnote 26.

[26] Formally let P_t denote the cumulative distribution function characterizing the measure $\mathbb{P}_t^u$ for $t \in \{1, ..., T\}$. P_t is said to have more weight in the tales than $\mathrm{P}_{t'}$, if $\int_{\underline{U}}^{\mathfrak{u}} \mathrm{P}_t(y) - \mathrm{P}_{t'}(y)\, dy \geq 0 \;\forall\, \mathfrak{u} \in [\underline{U}, \overline{U}]$.

possibility it permits to capture a difference in the weighting of expected welfare is by introducing a positive rate of pure time preference.

To my knowledge, the only consideration in the literature which is concerned with a relation between discounting and stationarity that is somewhat comparable to the one derived in theorem 14, is once more due to Epstein (1992, 16). Motivating models of recursive utility, he points out a contradiction between a disentanglement of risk aversion and intertemporal substitutability in a non-recursive model on the one hand, and the positiveness of the discount rate on the other. He concludes that a disentanglement is not possible, at least in a stationary setting. The preceding section has elaborated how such a disentanglement is possible in a non-stationary and in a certainty stationary setting. Theorem 14 confirms Epstein's (1992, 16) assertion, but with a very different interpretation. Having analyzed the reasons and consequences of an intrinsic timing preference, I suggest that a non-recursive evaluation is desirable (axiom A10). In consequence, a risk stationary decision maker in the sense of axiom A9 has to accept that he does not have the freedom to devaluate the future for sheer impatience, without violating any of the other axioms. Furthermore, theorem 14 together with theorem 13 show that, for a decision maker with a finite planning horizon, it is well possible to disentangle atemporal risk aversion from intertemporal substitutability, without violating any of the axioms. Moreover, also in the limit of an infinite planning horizon, a factor $\beta = 1$ does not necessarily imply that aggregate welfare diverges. Due to intertemporal risk aversion, an increase in uncertainty over time can still yield a finite evaluation of scenarios.[27] Of course, instead of accepting the consequences of theorem 14, the underlying axioms can be dropped. Since Chew & Epstein (1989, 110) have shown that under the assumption of axiom A10 the independence axiom can be replaced by a collection of much weaker axioms, it is not a promising candidate to give up in order to avoid the implication of a zero rate of pure time preference. If I had to drop an assumption, I would probably first abandon risk stationarity. In consequence, I had to allow for an anticipated change of preference over time.

Let me close the chapter by revisiting the important problem of global warming that has been discussed exemplarily in the introduction, as a motivation for my theoretical analysis in this dissertation. As Plambeck et al. (1997, 85) have pointed out, a reduction of the pure rate of time preference from 3%, as assumed in Nordhaus (1993), to 0% corresponding to $\beta = 1$, would result in an optimal abatement path that cuts emissions by

[27]Note however, that increasing uncertainty can also make the evaluation functional converge to zero. Preliminary analysis shows that convergence to finite non-zero evaluation are knife-edge in the assumptions on the probability distributions and their evolvement over time.

50% from the baseline to the year 2100, as opposed to 10% in the assessment of Nordhaus (1993). To the best of my knowledge, so far a zero rate of pure time preference has only been argued for in terms of moral consideration. Theorem 14 states formal axioms dealing with consistency aspects of evaluation under uncertainty, and shows that these alone suffice to call for a zero rate of pure time preference. In difference to the evaluations used in current climate models, however, the representation implied by theorem 14 goes along with an intertemporally risk averse decision maker. Therefore, uncertainty has a higher cost than in the above climate models, which apply the intertemporally risk neutral standard model when they consider uncertainty at all.[28] In consequence, an evaluation of global climate change under the assumptions of theorem 14, implies an additional preference for scenarios that give rise to a less uncertain future. Since uncertainty is likely to increase in the perturbation of the climate system, which increases with the amount of greenhouse gas emissions, a first conjecture is that the additional effect caused by intertemporal risk aversion in an evaluation in the sense of theorem 14, yields an even higher abatement recommendation than the one pointed out by Plambeck et al. (1997, 85). A closer analysis of this aspect constitutes an interesting area of future research.

10.5 Summary

I have extended the concept of intertemporal risk aversion to the *non-stationary multiperiod setting*. The general recursive and gaugable representation for preferences that are additive over time on certain consumption paths, and satisfy the von Neumann-Morgenstern axioms, has been developed. The axiomatic characterization, as well as the measures of intertemporal risk aversion, have been adapted to this framework. Different *stationarity* assumptions have been imposed on the general framework. These axioms offer an alternative to the standard stationarity axioms that rely on an infinite time horizon and a positive rate of pure time preference. First, certainty stationarity has been characterized and shown to imply the standard discount utility model on certain consumption paths. Then, aiming at a stationarity assumption that includes the generalized isoelastic model, I have worked out an axiom that implies constancy of the functions characterizing (atemporal) uncertainty aggregation. However, this axiom does not express the idea that the mere passage of time should not affect preference order-

[28]With the exception of the stylized simulation by Ha-Duong & Treich (2004) that features two possible damage states in a generalized isoelastic framework.

ings. A careful translation of the latter assumption to the evaluation of risky outcomes, implies that constancy of atemporal risk attitude is only supported for an infinite time horizon. Moreover, under a finite time horizon the corresponding axiom no longer admits for the whole class of generalized isoelastic evaluation rules. For risk stationary preferences, the measure of absolute intertemporal risk aversion is characterized by a single parameter, is constant in welfare, and increases over time.

I have explained that a decision maker using a recursive evaluation over temporal lotteries generally exhibits an *intrinsic preference for an early or late resolution of uncertainty*. The relation between such a preference for the timing of uncertainty resolution, and the characterizing functions of intertemporal substitutability and atemporal as well as intertemporal risk aversion has been given. I have analyzed the reasons for a non-trivial attitude with respect to the timing of uncertainty resolution in the recursive model, and have compared it to respective motivations found in the literature. As a result, I have suggested that *indifference to the timing of uncertainty resolution* is a desirable feature for a principled approach to choice under uncertainty. The corresponding preference representation has been stated, and I have worked out how the model allows to disentangle atemporal risk aversion from intertemporal substitutability in a non-recursive setting. Moreover, I have shown that timing indifference is compatible with the discount utility model on certain consumption paths. However, when adding stationarity of choice over risky outcomes to the assumptions, a devaluation of the future for reasons of sheer impatience is no longer allowed. Precisely, such a devaluation, corresponding to a strictly positive rate of *pure time preference* is only possible for intertemporally risk neutral decision makers, where the axiom of *risk* stationarity has no additional bite. However, when uncertainty is increasing over time, also an intertemporally risk averse decision maker values (expected) future welfare less than current welfare.

Chapter 11

Conclusions

11.1 Summary of Conceptual Contributions

This section summarizes the conceptual contributions of my dissertation. For a more detailed summary of the analysis carried out in each of the three parts, the reader is referred to the respective sections at the end of chapters 4, 7, and 10. Further implications and possible applications of the derived concepts and modeling frameworks are pointed out in section 11.2. Finally, I suggest different extensions of the study in section 11.3.

Structuring the contributions by parts, I start out with the exploration of the relation between the *weight given to future consumption* and service streams, and the *substitutability between environmental and produced service and consumption* streams in welfare. Under the assumption that produced consumption grows at a higher rate than environmental service and amenity streams, I show that a lower substitutability between the different classes of goods can imply a reduction of the weight given to future consumption and services. The result has two implications for the sustainability debate. First, the characterization of weak versus strong sustainability, resting upon a weak or strong limitedness in the substitutability between the two classes of goods, is directly connected to the weight given to future consumption and service streams. Second, a strength in sustainability in the above sense, can counteract a strength of sustainability in the sense of a higher weight given to consumption of future generations. Another contribution of the analysis in part I is, to point out that not only the value of the discount rate depends on the numeraire, but also the *form of discounting.* In particular, I derive that time consistent behavior in a growing world with limited substitutability in consumption can imply hyperbolic discount rates.

Part II of the dissertation introduces the concept of intertemporal risk aversion. On the one hand, the concept is designed to capture an important concern of the *precautionary principle*. In particular, an intertemporally risk averse decision maker exhibits a higher willingness to undergo preventive action in order to avoid a threat of harm, than does a decision maker employing the intertemporally additive expected utility model. On the other hand, the concept of intertemporal risk aversion sheds new light on the *disentanglement of atemporal risk aversion and intertemporal substitutability*. Such a disentanglement has been recognized in the literature to help in explaining several observed phenomena. The concept of intertemporal risk aversion elaborates that a specific difference between the characterizing functions of (atemporal) risk aversion and intertemporal substitutability has itself an interpretation of risk aversion. Quantitative measures of absolute and relative intertertemporal risk aversion are introduced. The corresponding concept of risk aversion extends naturally to the *multi-commodity* setting.

Part III extends the model framework for intertemporally risk averse decision making, and relates the concept to two other important aspects of decision making under uncertainty. Avoiding the assumption of a positive rate of pure time preference at the outset of the model, I derive an axiomatization of *stationary preferences* in a framework with a finite planning horizon. The standard interpretation of stationarity is that the mere passage of time does not influence preferences. I point out that for a finite planning horizon, an additional assumption is needed, in order to derive a stationary preference representation. This assumption is also implicit in the infinite horizon setting, but gains more bite under a finite planning horizon. In consequence, stationarity of risk attitude in the latter framework only allows for constant coefficients of absolute intertemporal risk aversion. In particular, the axiom excludes all specifications of the generalized isoelastic model, except for the case of logarithmic welfare.[1]

Moreover, part III analyzes the concept of an intrinsic *preference for the timing of uncertainty resolution*. I relate such a preference to the functions characterizing risk attitude and intertemporal substitutability in my representations. Connecting the concept to that of intertemporal risk aversion, I discuss the underlying intuition and draw the conclusion that an intrinsic preference for early or late resolution of uncertainty is likely to be undesired in a principled approach to decision making under uncertainty.

[1]Here, I refer to the axiom of risk stationarity II, capturing the idea that the mere passage of time should not change preferences. The word 'welfare' is used in the sense of the certainty additive Bernoulli utility function.

Eliminating the intrinsic timing preference from the model allows to depict intertemporal risk aversion and, thus, to disentangle atemporal risk aversion from intertemporal substitutability, in a non-recursive decision model. Such a non-recursive description of uncertainty constitutes a technical and conceptual simplification for the analysis of time and risk attitude. Two further insights are obtained by relating the concept of timing *indifference* to that of *stationarity*. First, I show that indifference to the timing of uncertainty resolution is compatible with the assumption of certainty stationarity and, thus, the standard discount utility approach to the evaluation of certain consumption paths. Second, for an intertemporally risk averse decision maker, accepting the von Neumann-Morgenstern axioms and time consistency, such an indifference is only compatible with a stationary risk attitude, if the pure rate of time preference is zero. Thus, accepting risk stationarity and the absence of an intrinsic preference for early or late resolution of uncertainty, the decision maker will not devaluate the future for reasons of pure time preference. However, he does so for reasons of increasing uncertainty over time.

11.2 Implications and Applications

The furthest reaching *implications* of this study are conveyed by the concept of *intertemporal risk aversion*. First, the concept *mediates between the advocates and the opponents of the precautionary principle*. On the one hand, the concept takes up the important concern of the principle's advocates regarding a higher willingness to undergo preventive action than that implied by a standard cost benefit assessment. On the other hand, it meets the requirements of the opponents by formalizing and, thus, sharpening this concern and reconciling it with standard assumptions underlying economic evaluation. This step allows to evaluate clearly and precisely in a more exhaustive way future threats to human welfare. In doing so, it also enables a more consistent and, thus, less disputable application of the precautionary principle, which gains increasing importance in international contracts and conventions, in particular in the field of the environment-economy interaction.

Second, the concept of intertemporal risk aversion implies a higher *welfare cost of volatility*. Since Lucas (1987), it is well known that the standard model in macroeconomics implies a strong 'bias' in favor of policy measures that foster additional growth at the expense of higher welfare volatility. If welfare volatility is caused by a stochastic process, the concept of intertemporal risk aversion explains that the above 'bias' is caused by the implicit assumption of intertemporal risk neutrality. Acknowledging intertemporal risk aversion implies an increased emphasis on considerations of welfare volatility in macroeconomic policy recommendations.

The third implication is a consequence of the fact that the concept of intertemporal risk aversion holds naturally in a *multi-commodity setting*. Currently, most studies on risk aversion only consider a single aggregate consumption good. This is partly due to the somewhat complicated and less satisfactory theory of multi-commodity risk aversion laid out by Kihlstrom & Mirman (1974). Intertemporal risk aversion characterizes risk attitude in a way that is independent of the amount of goods under observation. In consequence, an implication of the concept for economic modeling is to promote the explicit analysis of risk attitude in multi-commodity settings, including also more abstract situations where an a priori measure scale of the goods in terms of real numbers is not given.

My analysis on indifference with respect to the *timing of uncertainty resolution* has two immediate implications for decision makers in public policy. First, if a decision maker's willingness to substitute consumption into risky states is different from his propensity to substitute consumption into the certain future, I offer him a model where he no longer has to use an evaluation scheme that forces him to give up welfare for information that is of no use in the planning process. Second, if a decision maker accepts the axioms implying a zero rate of pure time preference, the long-term is gaining much more importance in model-based policy evaluations. While the certain future is treated equal to the present, the uncertain future gains the more importance, the more the decision maker can know about it. For a particular application of this reasoning, see the last paragraph of this section.

The insight that a strong *sustainability preference* can imply a lower *weight for future consumption* streams than a weak sustainability preference, mainly has implications for the question how to depict different concepts of sustainability in economic modeling. Policy implication rather emerge in more specific *applications*. For example, the insight that *limited substitutability in consumption* can influence the effective discount rate, is relevant for climate change evaluation. The corresponding models generally depict an aggregate of consumption and employ a corresponding real discount rate. Yet, global climate change is predicted to affect future flows of environmental services and produced consumption significantly. Therefore, a non-constant time behavior of discount rates as pointed out in the study, can be of particular importance. For a quantification of this effect, however, an extension of the model is suggested in the next section. Apart from evaluation studies, the insight that limited substitutability in consumption can affect the *form of discounting*, is as well of interest for experimental economics. Here, the topic of hyperbolic discounting is an active field of research. Applying the model to experimental settings, it is a promising exercise to examine to what extent the reasoning on limited substitutability can contribute to an understanding of non-constant discount rates observed in the laboratory.

Applications for the concept of *intertemporal risk aversion and the respective modeling frameworks* derived in parts II and III are numerous. The concept can be applied to any field of economic evaluation, decision making or modeling, where time and uncertainty play an essential role, reaching from classical resource extraction problems to asset pricing in stock markets. As an application of particular conceptual interest, I consider a closer analysis of the paradox discovered by Rabin (2000) on the relation between *risk aversion in the small and in the large* at different welfare levels. The latter has recently been extended to non-expected utility theory by Safra & Segal (2005). As the concept of intertemporal risk aversion allows to detach the description of risk aversion from the curvature of welfare, it constitutes a natural candidate to reanalyze the paradox.

The modeling frameworks in this dissertation have been motivated from the viewpoint of a principled and, thus, prescriptive approach to choice under uncertainty. I consider it an important challenge to *test* how the derived representations perform in describing observed phenomena. Of particular interest is the question, whether the developed framework that allows to disentangle intertemporal substitutability from risk aversion under indifference to the timing of uncertainty resolution, can quantitatively outperform the generalized isoelastic model, which is the current work horse for this task. For reasons of data availability and the amount of research already performed in the field, the stock market is a good place to start such a comparison. The model's ability to explain the *equity premium puzzle* constitutes an interesting topic to investigate.

Another important application of the derived representational framework for intertemporal risk aversion is to take up once more the problem of *climate change* that has been discussed in the introduction as a motivation for my theoretical analysis. I have pointed out that a zero rate of pure time preference in climate change evaluation models significantly modifies the recommendations for an optimal greenhouse gas abatement path (Toth 1995, Plambeck et al. 1997). Founding such a zero rate of pure time preference on my analysis of intertemporal risk aversion, rather than on moral considerations, adds a new aspect to the evaluation. In contrast to the modeling approach of the above authors, an evaluation featuring intertemporal risk aversion implies discounting for reasons of uncertainty. In general, increasing uncertainty over time brings about a decreasing weight given to future welfare and, thus, future consumption and service streams. However, if the amount of uncertainty differs for the different abatement scenarios, those that render future consumption less uncertain obtain a higher weight. Since uncertainty is likely to increase in the perturbation of the climate system, which increases with the amount of greenhouse gas emissions, a first conjecture is that the additional effect caused by intertemporal risk aversion yields an even higher abatement recommendation than the one pointed out by the above studies. I plan to carry out a quantitative assessment of the effect in the future.

11.3 Extensions

Apart from these direct applications pointed out in the preceding section, several extensions of the models and concepts discussed in this study seem worthwhile. I suggest some of the standard extensions and others that appear conceptually most interesting. Starting with the model employed in part I of this dissertation, two extensions that 'destylize' the model stand to reason when the model is to be applied quantitatively.[2] The first extension would be to drop the assumption of a *constant elasticity of substitution* between the two classes of goods.[3] The second would be to model and estimate precisely the *supply side* of the model, which so far is pressed into the simple assumption of differing growth rates in consumption. Furthermore, an integration of the models in parts I and II stands to reason. In such a model, the *combined effect of substitutability between goods, between periods and between risk states* on the *discount rate* can be analyzed. A final extension which I want to point out, is based on the insight that the marginal utility propagators used in the model can also be defined for more general *non-conservative preference fields*, which do not possess a closed form representation in terms of utility. Then, 'value' development over time can still be described in terms of their generators, i.e. the individual discount rates can replace the concept of utility. Thus, a comparable analysis to that in part I can be carried out in a much more general setting.

Turning to extensions of the modeling frameworks for the concept of intertemporal risk aversion, I first want to mention those which I consider primarily technical. One is to drop the assumption of *additive separability* over time. While the same axiomatic characterization of intertemporal risk aversion holds, it is interesting to work out the functional representation in the generalized recursive framework and to compare it to the one derived in my setting. Another extension is to translate the model into a *continuous*

[2]See section 11.2. While an application to long-term evaluation studies would make both extensions necessary, an application to the analysis of laboratory experiments allows to control for the 'supply' side.

[3]Of course, at the same time the assumption that there are only *two* different classes could be dropped.

time setting. While this step is immediate for the non-recursive representation, the extension for the general setting can work along the lines of Duffie & Epstein (1992). Finally, a comparable, but slightly more conceptual extension is to allow for *history dependence* of preferences. While this step complicates the representations in terms of Bernoulli utility functions, it should not limit the definition and representation of intertemporal risk aversion.

A conceptually very promising extension is to combine the reasoning on intertemporal risk aversion with concepts of *ambiguity*. In particular, a combination of my time and uncertainty structure with Ghirardato et al.'s (2004) axiomatization of ambiguity attitude can bring about an attractive model for choice under uncertainty, and a concept of 'intertemporal ambiguity'. Another possible extension picks up a point that has been judged an application for the specific problem of global warming and suggests to analyze it from a more general perspective. The object of investigation is the relation between the *form of discounting* and the specifications of intertemporal risk aversion, the probability distribution over uncertain outcomes, and its evolution over time. In particular the question arises, under what assumptions on intertemporal risk aversion and uncertainty evolvement, a welfare representation with a zero rate of pure time preference converges in an infinite time horizon. Finally, a useful extension of my setting is to analyze the *interpersonal aggregation* of welfare for intertemporally risk averse decision makers. To this end, a comprising axiomatic framework that develops assumptions on the interpersonal comparability of risk attitude and welfare has to be elaborated.

Summing up, I have suggested a principled approach to long-term evaluation and decision making. Several related insights have been derived, which shall provoke and support a discussion on how to treat the, mostly uncertain, long run in environmental evaluation and economic modeling. The study has opened up several alleys of future research, and I hope that these lead a way to future answers.

Appendix A

Proofs and Calculations for Part I

A.1 Calculations for Chapter 2

Derivation of the finite time propagator, of marginal utility:
Using the multiplicative structure of the propagator the derivation of $D_i^{\mathsf{x}}(t, t_0)$ from $D_i^{\mathsf{x}}(t + dt, t)$ is straightforward:

$$\begin{aligned}
D_i^{\mathsf{x}}(t+dt, t_0) &= \frac{\frac{\partial U}{\partial x_i}(t+dt)}{\frac{\partial U}{\partial x_i}(t_0)} = \frac{\frac{\partial U}{\partial x_i}(t+dt)}{\frac{\partial U}{\partial x_i}(t)} \frac{\frac{\partial U}{\partial x_i}(t)}{\frac{\partial U}{\partial x_i}(t_0)} \\
&= D_i^{\mathsf{x}}(t+dt, t) D_i^{\mathsf{x}}(t, t_0) \\
\Rightarrow D_i^{\mathsf{x}}(t+dt, t_0) - D_i^{\mathsf{x}}(t, t_0) &= \underbrace{\left(D_i^{\mathsf{x}}(t+dt, t) - 1\right)} \; D_i^{\mathsf{x}}(t, t_0) \\
&\qquad\qquad \| \\
&= \overbrace{-\delta_i(x(t), \dot{x}(t), t)\, dt} \; D_i^{\mathsf{x}}(t, t_0) \\
\Rightarrow \frac{D_i^{\mathsf{x}}(t+dt, t_0) - D_i^{\mathsf{x}}(t, t_0)}{dt} &= -\delta_i(x(t), \dot{x}(t), t)\, D_i^{\mathsf{x}}(t, t_0) \\
\Rightarrow \frac{d}{dt} D_i^{\mathsf{x}}(t, t_0) &= -\delta_i(x(t), \dot{x}(t), t)\, D_i^{\mathsf{x}}(t, t_0) \\
\Rightarrow \frac{d}{dt} \ln D_i^{\mathsf{x}}(t, t_0) &= -\delta_i(x(t), \dot{x}(t), t) \\
\Rightarrow D_i^{\mathsf{x}}(t, t_0) &= a\, e^{\int_{t_0}^{t} -\delta_i(x(t'), \dot{x}(t'), t')\, dt'} \; .
\end{aligned}$$

Because of $D_i^{\mathsf{x}}(t, t) = 1$ the integration constant a must be equal to 1.

Calculation of the social discount rate for $U(x_1, x_2, t) = [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}} e^{-\rho t}$ **:**

The derivatives needed for the computation of δ_1 are for $s \notin \{0,1\}$:

$$\frac{\partial U}{\partial x_1} = a_1 u_1(x_1)^{s-1} u_1'(x_1) \Big[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s\Big]^{\frac{1}{s}-1} e^{-\rho t} \;,$$

$$\begin{aligned}\frac{\partial^2 U}{\partial x_1^2} &= \Big(a_1 u_1(x_1)^{s-1} u_1''(x_1) - (1-s) a_1 u_1(x_1)^{s-2} u_1'(x_1)^2\Big) \\ &\quad \cdot \Big[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s\Big]^{\frac{1}{s}-1} \cdot e^{-\rho t} \\ &\quad + (1-s)\Big(a_1 u_1(x_1)^{s-1}\Big)^2 u_1'(x_1)^2 \Big[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s\Big]^{\frac{1}{s}-2} e^{-\rho t} \quad \text{and}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 U}{\partial x_1 \partial x_2} &= (1-s)\Big(a_1 u_1(x_1) a_2 u_2(x_2)\Big)^{s-1} u_1'(x_1) u_2'(x_2) \\ &\quad \cdot \Big[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s\Big]^{\frac{1}{s}-2} e^{-\rho t} \;.\end{aligned}$$

Inserting these into equation (3.1) yields:

$$\begin{aligned}\delta_1(t) &= \rho - \frac{(a_1 u_1(x_1)^{s-1} u_1''(x_1) - (1-s) a_1 u_1(x_1)^{s-2} u_1'(x_1)^2)\,[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1-s}{s}}}{a_1 u_1(x_1)^{s-1} u_1'(x_1) [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}} \\ &\quad \cdot \dot{x}_1 - \frac{(1-s)(a_1 u_1(x_1)^{s-1})^2 u_1'(x_1)^2 [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-2}}{a_1 u_1(x_1)^{s-1} u_1'(x_1) [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}} \,\dot{x}_1 \\ &\quad - \frac{(1-s)\,(a_1 u_1(x_1) a_2 u_2(x_2))^{s-1}\, u_1'(x_1) u_2'(x_2) [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-2}}{a_1 u_1(x_1)^{s-1} u_1'(x_1) [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}} \,\dot{x}_2 \\ &= \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\,\dot{x}_1 + (1-s) u_1(x_1)^{-1} u_1'(x_1)\,\dot{x}_1 \\ &\quad -(1-s)\frac{a_1 u_1(x_1)^{s-1} u_1'(x_1)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\,\dot{x}_1 - (1-s)\frac{a_2 u_2(x_2)^{s-1} u_2'(x_2)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\,\dot{x}_2 \\ &= \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\,\dot{x}_1 \\ &\quad +(1-s)\frac{u_1(x_1)^{-1} u_1'(x_1)(a_1 u_1(x_1)^s + a_2 u_2(x_2)^s) - a_1 u_1(x_1)^{s-1} u_1'(x_1)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\,\dot{x}_1 \\ &\quad -(1-s)\frac{a_2 u_2(x_2)^{s-1} u_2'(x_2)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\,\dot{x}_2 \\ &= \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\,\dot{x}_1 + (1-s)\frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\;\frac{u_1'(x_1)}{u_1(x_1)}\,\dot{x}_1 \\ &\quad -(1-s)\frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\;\frac{u_2'(x_2)}{u_2(x_2)}\,\dot{x}_2 \;.\end{aligned}$$

Which brings about equation (3.2):

$$\delta_1(t) = \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\dot{x}_1 - (1-s)\frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\left(\frac{u_2'(x_2)}{u_2(x_2)}\dot{x}_2 - \frac{u_1'(x_1)}{u_1(x_1)}\dot{x}_1\right).$$

For $s = 1$ with $\frac{\partial^2 U}{\partial x_1 \partial x_2} = 0$ and $1 - s = 0$ it is easily observed that the same equation has to hold. For the case $s = 1$ it is $u(x_1, x_2) = [a_1 x_1^s + a_2 x_2^s]^{1/s}$ (see footnote 6 on page 27). The derivatives needed for the computation of δ_1 are

$$\begin{aligned}
\frac{\partial U}{\partial x_1} &= a_1 u_1(x_1)^{a_1-1} u_1'(x_1) u_2(x_2)^{a_2} e^{-\rho t}\ , \\
\frac{\partial^2 U}{\partial x_1^2} &= a_1(a_1-1) u_1(x_1)^{a_1-2} {u_1'(x_1)}^2 u_2(x_2)^{a_2} e^{-\rho t} \\
&\quad + a_1 u_1(x_1)^{a_1-1} u_1''(x_1) u_2(x_2)^{a_2} e^{-\rho t} \qquad \text{and} \\
\frac{\partial^2 U}{\partial x_1 \partial x_2} &= a_1 u_1(x_1)^{a_1-2} u_1'(x_1) a_2 u_2(x_2)^{a_2-1} u_2'(x_2) e^{-\rho t}\ .
\end{aligned}$$

These derivatives deliver the social discount rate

$$\delta_1(t) = \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\dot{x}_1 - a_2\left(\frac{u_2'(x_2)}{a_2 u_2(x_2)}\dot{x}_2 - \frac{u_1'(x_1)}{a_1 u_1(x_1)}\dot{x}_1\right)$$

which coincides with equation (3.2) for $s = 0$ as $a_1 + a_2 = 1$.

A.2 Calculations and Proofs for Chapter 3

Some of the proofs make use of the following

Transformation of $V\!\!\!/sp^s$ and $V\!\!\!/se^s$:

First note that the following relation holds:

$$\begin{aligned}
\frac{d \ln x_i(t)}{dt} &= \frac{\dot{x}_i(t)}{x_i(t)} \\
\Rightarrow d \ln x_i(t)\, dt &= \hat{x}_i(t)\, dt \\
\Rightarrow \quad \ln x_i(t) &= \int_0^t \hat{x}_i(t')\, dt' + c \\
\Rightarrow \quad x_i(t) &= x_i(0) e^{\int_0^t \hat{x}_i(t')\, dt'} \\
\Rightarrow \quad x_i(t)^s &= x_i(0)^s e^{s \int_0^t \hat{x}_i(t')\, dt'}\ .
\end{aligned}$$

Therefore the term Vsp^s can be transformed as follows:

$$\begin{aligned} \mathit{Vsp}^s(x_1,x_2) &= \frac{a_2 x_2(t)^s}{a_1 x_1(t)^s + a_2 x_2(t)^s} \\ &= \frac{a_2 x_2(0)^s e^{s\int_0^t \hat{x}_2(t')\,dt'}}{a_1 x_1(0)^s e^{s\int_0^t \hat{x}_1(t')\,dt'} + a_2 x_2(0)^s e^{s\int_0^t \hat{x}_2(t')\,dt'}} \\ &= \frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s}\frac{e^{s\int_0^t \hat{x}_1(t')\,dt'}}{e^{s\int_0^t \hat{x}_2(t')\,dt'}} + 1} = \frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t')-\hat{x}_1(t')\,dt'} + 1} \,. \end{aligned} \tag{A.1}$$

By switching the indices in equation (A.1) one finds the corresponding expression for the value share of the environmental good:

$$\mathit{Vse}^s(x_1,x_2) = \frac{a_1 x_1(t)^s}{a_1 x_1(t)^s + a_2 x_2(t)^s} = \frac{1}{\frac{a_2 x_2(0)^s}{a_1 x_1(0)^s} e^{s\int_0^t \hat{x}_2(t')-\hat{x}_1(t')\,dt'} + 1} \,.$$

Proof of Proposition 1: The proposition is derived in the text. □

Proof of Proposition 2: Note that all terms in equations (3.3) and (3.6) are positive. *First* I show that Vsp^s is strictly increasing. As derived above equation (A.1) holds:

$$\mathit{Vsp}^s = \frac{a_2 x_2(t)^s}{a_1 x_1(t)^s + a_2 x_2(t)^s} = \frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t')-\hat{x}_1(t')\,dt'} + 1} \,.$$

From $\hat{x}_2(t) - \hat{x}_1(t) > 0 \,\forall\, t$ and $s > 0$ it follows that the expression $\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t')-\hat{x}_1(t')\,dt'}$ is strictly falling in time. Therefore the value share of the produced consumption stream Vsp^s is strictly increasing over time.

Second, such an increasing Vsp^s implies that the second term in the social discount rate for the environmental amenity stream $(1-s)\mathit{Vsp}^s\,(x_1(t), x_2(t))\,(\hat{x}_2 - \hat{x}_1)$ is increasing in a steady state. As this term is subtracted from the constant rate of pure time preference, the social discount rate for the first commodity class $\delta_1(t)$ *declines* in a steady state.

Third, a strictly increasing term Vsp^s implies a strictly decreasing value share of the environmental amenity stream $\mathit{Vse}^s = 1 - \mathit{Vsp}^s$. Such a strictly decreasing term Vse^s implies that the the expression $(1-s)\mathit{Vse}^s\,(x_1(t), x_2(t))\,(\hat{x}_2 - \hat{x}_1)$ strictly decreases in a steady state. As this expression is added to the constant rate of pure time preference to yield the social discount rate for the produced consumption stream, the social discount rate $\delta_2(t)$ *declines* as well in a steady state.

Finally, if there exist $\epsilon > 0$ and $t^* \in [0, \infty)$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all $t \geq t^*$, which under assumption 1 is in particular satisfied in a steady state, the expression $\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t')-\hat{x}_1(t')\,dt'}$ falls to zero and the value share Vsp^s grows to unity. Therefore in a steady state the discount rate δ_1 monotonously falls to $\delta_1 = \rho - (1-s)(\hat{x}_2 - \hat{x}_1)$

for $t \to \infty$. At the same time the value share of the environmental amenity stream Vse^s falls to zero implying that the social discount rate for the produced consumption stream falls to $\delta_2 = \rho$. □

Proof of Proposition 3: *First* I show that Vsp^s is strictly decreasing. As derived above equation (A.1) holds:

$$\mathit{Vsp}^s = \frac{a_2 x_2(t)^s}{a_1 x_1(t)^s + a_2 x_2(t)^s} = \frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s \int_0^t \hat{x}_2(t') - \hat{x}_1(t')\, dt'} + 1} \, .$$

But then from $\hat{x}_2(t) - \hat{x}_1(t) > 0 \, \forall t$ and $s < 0$ it follows that the expression $\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s \int_0^t \hat{x}_2(t') - \hat{x}_1(t')\, dt'}$ is strictly increasing in time. Therefore that value share of the produced consumption stream Vsp^s is strictly decreasing over time.

Second, such a decreasing Vsp^s implies that the second term in the social discount rate for the environmental amenity stream $(1-s)\mathit{Vsp}^s\,(x_1(t), x_2(t))\,(\hat{x}_2 - \hat{x}_1)$ is decreasing in a steady state. As this term is subtracted from the constant rate of pure time preference, the social discount rate for the first commodity class $\delta_1(t)$ *grows* in a steady state.

Third, a strictly decreasing term Vsp^s implies a strictly increasing value share of the environmental service stream $\mathit{Vse}^s = 1 - \mathit{Vsp}^s$. Such a strictly increasing term Vse^s implies that the the expression $(1-s)\mathit{Vsp}^s\,(x_1(t), x_2(t))\,(\hat{x}_2 - \hat{x}_1)$ strictly increases in a steady state. As this expression is added to the constant rate of pure time preference to yield the social discount rate for the produced consumption stream, the social discount rate $\delta_2(t)$ *grows* as well in a steady state.

Finally if there exist $\epsilon > 0$ and $t^* \in [0, \infty)$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all $t \geq t^*$, which in particular is satisfied under assumption 1 in a steady state, the expression $\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s \int_0^t \hat{x}_2(t') - \hat{x}_1(t')\, dt'}$ grows without bounds and the value share Vsp^s falls to zero. Therefore in a steady state the discount rate δ_1 monotonously grows to $\delta_1 = \rho$ for $t \to \infty$. At the same time the value share of the environmental amenity stream Vse^s grows to one, implying that the discount rate for the produced consumption stream grows to $\delta_2 = \rho + (1-s)(\hat{x}_2 - \hat{x}_1)$. □

Proof of Proposition 4: The result for the steady state follows immediately from propositions 2 and 3. For the social discount rate of the environmental amenity stream the propositions establish the relation $\lim\limits_{t\to\infty} \delta_1^{0<s<1}(t) = \delta_1 = \rho - (1-s)(\hat{x}_2 - \hat{x}_1) < \rho = \lim\limits_{t\to\infty} \delta_1^{s<0}(t)$. For the social discount rate of the produced consumption stream the propositions establish the relation $\lim\limits_{t\to\infty} \delta_2^{0<s<1}(t) = \rho < \rho + (1-s)(\hat{x}_2 - \hat{x}_1) = \lim\limits_{t\to\infty} \delta_2^{s<0}(t)$.

The proof for the statement assuming only the existence of $\epsilon > 0$ and $t^* \in [0, \infty)$ with $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all $t \geq t^*$ is as follows. Consider the long run social discount rate for the environmental amenity stream. In the proof of proposition 2 I have shown that, if there exist $\epsilon > 0$ and $t^* \in [0, \infty)$ with $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$, then the term $V\!sp^{0<s<1}$ monotonously grows to unity as $t \to \infty$. In particular there has to exist $t_1 \in [0, \infty)$ such that $V\!sp^{0<s<1} > \frac{2}{3} \forall t > t_1$, implying

$$\begin{aligned} &(1-s)\; V\!sp^{0<s<1}\; (\hat{x}_2(t) - \hat{x}_1(t)) > (1-s)\, \frac{2}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ \Rightarrow\; &\delta_1^{0<s<1}(t) = \rho - (1-s)\; V\!sp^{0<s<1}(t)\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &\qquad < \rho - (1-s)\, \frac{2}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \end{aligned}$$

for all $t > t_1$. Similarly the fact that for $s < 0$ the proof of proposition 3 has shown that $V\!sp^{s<0}$ monotonously falls to zero as $t \to \infty$ implies the existence of t_2 such that $V\!sp^{s<0} < \frac{1}{3}$. Then for the social discount rate of the in the strong sustainability scenario it follows

$$\begin{aligned} &(1-s)\; V\!sp^{s<0}\; (\hat{x}_2(t) - \hat{x}_1(t)) < (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ \Rightarrow\; &\delta_1^{s<0}(t) = \rho - (1-s)\; V\!sp^{s<0}(t)\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &\qquad > \rho - (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \end{aligned}$$

for all $t > t_2$. Setting $t_3 = \max\{t_1, t_2\}$ I find

$$\begin{aligned} \delta_1^{s<0}(t) &> \rho - (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &> \rho - (1-s)\, \frac{2}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &> \delta_1^{0<s<1}(t) \end{aligned}$$

for all $t > t_3$. Analogously one derives for the social discount rate of the produced consumption stream the existence of $t_1' \in [0, \infty)$ such that for $0 < s < 1$ it holds

$$\begin{aligned} \delta_2^{0<s<1}(t) &= \rho + (1-s)\; V\!se^{0<s<1}(t)\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &< \rho + (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \end{aligned}$$

for all $t > t_1'$ (as $V\!se^s$ goes to zero), and the existence of $t_2' \in [0, \infty)$ such that for $s < 0$ it holds

$$\begin{aligned} \delta_2^{s<0}(t) &= \rho + (1-s)\; V\!se^{s<0}(t)\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &> \rho + (1-s)\, \frac{2}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \end{aligned}$$

for all $t > t_2'$ (as Vse^s grows to unity). Then setting $t_3' = \max\{t_1', t_2'\}$ delivers the relation

$$\begin{aligned}\delta_1^{s<0}(t) &> \rho + (1-s)\,\frac{2}{3}\,\left(\hat{x}_2(t) - \hat{x}_1(t)\right) \\ &> \rho + (1-s)\,\frac{1}{3}\,\left(\hat{x}_2(t) - \hat{x}_1(t)\right) \\ &> \delta_1^{0<s<1}(t)\end{aligned}$$

for all $t > t_3'$. Setting $\bar{t} = \max\{t_3, t_3'\}$ yields the statement of the proposition. □

Proof of Proposition 5: The proof of proposition 4 brings about the existence of $\bar{t}\,'$ and $\epsilon > 0$ such that

$$\begin{aligned} & \delta_i^{s<0}(t) - \delta_i^{0<s<1}(t) && > \epsilon && \text{for all } t > \bar{t}\,' \\ \Leftrightarrow\ & \exp\left(\int_{\bar{t}'}^{t} \delta_i^{s<0}(t) - \delta_i^{0<s<1}(t)\,dt'\right) && > \exp\left(\int_{\bar{t}'}^{t} \epsilon\,dt'\right) && \text{for all } t > \bar{t}\,' \\ \Leftrightarrow\ & \frac{D_i^{\chi^{0<s<1}}(t, t_{\bar{t}'})}{D_i^{\chi^{s<0}}(t, t_{\bar{t}'})} && > \exp\left(\int_{\bar{t}'}^{t} \epsilon\,dt'\right) && \text{for all } t > \bar{t}\,'\,. \end{aligned}$$

Therefore, for any $t_0 \in [0,\infty)$ the following relation has to hold for some constant $C \in \mathbb{R}_{++}$:

$$\begin{aligned}\frac{D_i^{\chi^{0<s<1}}(t,t_0)}{D_i^{\chi^{s<0}}(t,t_0)} &= \frac{D_i^{\chi^{0<s<1}}(t_{\bar{t}'},t_0)}{D_i^{\chi^{s<0}}(t_{\bar{t}'},t_0)}\ \frac{D_i^{\chi^{0<s<1}}(t,t_{\bar{t}'})}{D_i^{\chi^{s<0}}(t,t_{\bar{t}'})} \\ &> \quad C \quad \exp\left(\int_{\bar{t}'}^{t} \epsilon\,dt'\right)\ . \end{aligned} \tag{A.2}$$

As the right hand side of equation (A.2) grows to infinity for $t \to \infty$ the left hand side in particular grows bigger than one. Hence it exists $\bar{t}$ such that

$$D_i^{\chi^{0<s<1}}(t,t_0) > D_i^{\chi^{s<0}}(t,t_0) \quad \text{for all } t > \bar{t}\,.$$

□

Appendix B

Proofs for Part II

B.1 Notation and lemma

For convenience I denote the group of nondegenerate affine transformations by $\mathsf{A} = \{\mathfrak{a} : \mathbb{R} \to \mathbb{R} : \mathfrak{a}(z) = a\,z + b\,,\; a, b \in \mathbb{R}, a \neq 0\}$ with elements $\mathfrak{a} \in \mathsf{A}$ and similarly the group of positive affine transformations with elements $\mathfrak{a}^+$ by $\mathsf{A}^+ = \{\mathfrak{a}^+ : \mathbb{R} \to \mathbb{R} : \mathfrak{a}^+(z) = a\,z + b\,,\; a, b \in \mathbb{R}, a > 0\}$. This notation will also be introduced at a later point (page 74) in the main text.
Note: For $\mathfrak{a} \in \mathsf{A}$ it holds that $\mathcal{M}^{\mathfrak{a}f}(u,p) = f^{-1}\mathfrak{a}^{-1} \int_X \mathfrak{a} f u \; dp = f^{-1}\mathfrak{a}^{-1}\mathfrak{a} \int_X f u \; dp = f^{-1}\left[\int_X f u \; dp\right] = \mathcal{M}^f(u,p)$ (the composition sign $\circ$ has been supressed).
Some proofs in appendix B.3 will make use of the following lemma.

Lemma 0: If the tuple (u, f) represents $\succeq$ in the sense of proposition 7, then so does the tuple $(s \circ u, f \circ s^{-1})$ for any $s : U \to \mathbb{R}$ strictly increasing and continuous.

Proof of lemma 0: The second tuple stands for the representation $sf^{-1}\left[\int_X (fs^{-1})(su) \; dp\right] = sf^{-1}\left[\int_X fu \; dp\right]$. The latter is a strictly increasing transformation of the representation $\mathcal{M}^f(p,u)$ for $\succeq$ and hence a representation for $\succeq$ itself. Moreover $s \circ u$ and $f \circ s^{-1}$ are continuous and the latter is strictly monotonic.[1] □

[1]Continuity of s^{-1} follows from the fact that the inverse of a strictly monotonic function on an interval is continuous (Heuser 1988, 231).

B.2 Proofs for Chapter 5

Proof of proposition 6: "⇒": Let $\mathcal{M}^f$ satisfy equation (5.4). Then for some given function u it is

$$\begin{aligned} \mathcal{M}^f(p,u) &< \mathrm{E}(p,u) \qquad \forall\, p \in P_u^{th} \\ \Rightarrow f^{-1}\left[p(\bar{x})f[u(\bar{x})] + p(\underline{x})f[u(\underline{x})]\right] &< p(\bar{x})u(\bar{x}) + p(\underline{x})u(\underline{x}) \qquad \text{(B.1)} \end{aligned}$$

for all $p \in P^s$ such $p(\bar{x}), p(\underline{x}) > 0$ and $p(\bar{x}) + p(\underline{x}) = 1$. Moreover equation (B.1) has to hold for all elements $\underline{x}, \bar{x} \in X$ that satisfy the condition $u(\bar{x}) > u(\underline{x})$. Define $\underline{\mathrm{p}} = p(\underline{x})$ and $\bar{\mathrm{p}} = p(\bar{x})$. Then the requirement $p(\bar{x}), p(\underline{x}) > 0$ and $p(\bar{x}) + p(\underline{x}) = 1$ translates into $\bar{\mathrm{p}}, \underline{\mathrm{p}} \in (0,1)$, $\bar{\mathrm{p}} + \underline{\mathrm{p}} = 1$. Similarly define $\underline{\mathrm{u}} = u(\underline{x})$ and $\bar{\mathrm{u}} = u(\bar{x})$. Then $u(\bar{x}) > u(\underline{x})$ translates into $\bar{\mathrm{u}} > \underline{\mathrm{u}}$. With these definitions (B.1) can be written as

$$\Leftrightarrow f^{-1}\left[\bar{\mathrm{p}}\, f[\,\bar{\mathrm{u}}\,] + \underline{\mathrm{p}} f[\,\underline{\mathrm{u}}\,]\right] < \bar{\mathrm{p}}\bar{\mathrm{u}} + \underline{\mathrm{p}}\underline{\mathrm{u}} \qquad \text{(B.2)}$$

and has to hold for all $\bar{\mathrm{p}}, \underline{\mathrm{p}} \in (0,1)$ with $\bar{\mathrm{p}} + \underline{\mathrm{p}} = 1$ and for all $\bar{\mathrm{u}}, \underline{\mathrm{u}} \in U$ with $\bar{\mathrm{u}} > \underline{\mathrm{u}}$. Note that the symmetry in (B.2) implies that the condition $\bar{\mathrm{u}} > \underline{\mathrm{u}}$ can be replaced by the condition $\bar{\mathrm{u}} \neq \underline{\mathrm{u}}$. But for an increasing function f equation (B.2) is equivalen to

$$\Leftrightarrow \bar{\mathrm{p}}\, f[\,\bar{\mathrm{u}}\,] + \underline{\mathrm{p}} f[\,\underline{\mathrm{u}}\,] < f[\,\bar{\mathrm{p}}\bar{\mathrm{u}} + \underline{\mathrm{p}}\underline{\mathrm{u}}\,]$$

and has to hold for all $\bar{\mathrm{p}}, \underline{\mathrm{p}} \in (0,1), \bar{\mathrm{p}} + \underline{\mathrm{p}} = 1$ and for all $\bar{\mathrm{u}}, \underline{\mathrm{u}} \in U$ with $\bar{\mathrm{u}} \neq \underline{\mathrm{u}}$. But this is just the definition of concavity of f on U. Similarly for a decreasing function f the relation

$$\Leftrightarrow \bar{\mathrm{p}}\, f[\,\bar{\mathrm{u}}\,] + \underline{\mathrm{p}} f[\,\underline{\mathrm{u}}\,] > f[\,\bar{\mathrm{p}}\bar{\mathrm{u}} + \underline{\mathrm{p}}\underline{\mathrm{u}}\,]$$

has to hold and defines convexity of f on U. As a strictly monotonic function is either strictly increasing or strictly decreasing the first assertion in the proposition follows. The second part for the uncertainty aggregation rule $\mathcal{M}^\alpha$ follows immediately from Hardy et al. (1964, 26) as has already been mentioned in the text. Alternatively verify that $f(z) = z^\alpha$ is strictly decreasing and convex for $\alpha < 0$ and strictly increasing and concave for $\alpha \in (0,1)$ but strictly increasing and convex for $\alpha > 1$. The case $\alpha = 0$ was defined by limit. As it holds that $\lim_{\alpha\to 0} \mathcal{M}^{f(z)=z^\alpha} = \mathcal{M}^{\ln}$ and the natural logarithm is strictly increasing and concave the uncertainty aggregation rule corresponding to the geometric mean and $\alpha = 0$ is precautionary as well.
"⇐": Take any $p \in P_u^{th}$. Going backwards the steps in in the first part of the proof immediately gives that strict concavity for an increasing function f and strict convexity for a decreasing f yield equations (B.1) and (B.2). Thus the evaluation of any threat of harm lottery by means of the uncertainty aggregation rule $\mathcal{M}^f$ renders a lower evaluation than does expected value. □

B.3 Proofs for Chapter 6

Proof of theorem 1: As X is a compact metric space it is Polish and, thus, separable. Therefore the theorem follows immediately from theorem 3 in Grandmont (1972). □

Proof of proposition 7: "⇒": By axioms A1-A3 theorem 1 gives the existence of the representation (u^0, id) with u^0 as in theorem 1. Because u of proposition 7 and u^0 of theorem 1 are in $B_{\succeq}$ it is $u^0(x^1) \geq u^0(x^2) \Leftrightarrow \delta_{x^1} \succeq \delta_{x^2} \Leftrightarrow u(x^1) \geq u(x^2)$ for all $x^1, x^2 \in X$. Therefore a strictly increasing transformation s *exists such that* $u = s \circ u^0$. Let $U^0 \equiv \text{range}(u^0)$ and $U \equiv \text{range}(u)$. To see that continuity of u and u^0 bring about continuity of $s : U^0 \to U$ it is enough to find that the preimage of *any closed subset* $A \subset U$ under s is closed. As u is continuous the preimage of A under u, $B = u^{-1}(A)$, is closed. Moreover a closed subset of a compact space B is compact and the image of a compact set under the continuous function u^0 is compact (Schofield 2003, 111). In consequence the resulting image $u^0(B)$, which is the sought for *preimage of* A *under* s,[2] *is closed. Hence,* s *is continuous.* Therefore by lemma 0 the tuple $(s \circ u^0, s^{-1})$ represents $\succeq$. Let $f^+ \equiv s^{-1}$, then f^+ is strictly increasing and continuous and (u, f^+) represent $\succeq$ in the sense of proposition 7.
"⇐": First let f be strictly increasing and (u, f) represent $\succeq$ in the sense of proposition 7. Applying lemma 0 with $s = f^{-1}$ strictly increasing and continuous (see footnote 1) shows that $(f^{-1} \circ u, \text{id})$ also represents $\succeq$ in this sense. But than $u^0 \equiv f^{-1} \circ u$ represents $\succeq$ in the sense of theorem 1. Therefore the latter assures that A1-A3 are satisfied. For f strictly decreasing note that $\mathcal{M}^f = \mathcal{M}^{-f}$ and hence the above reasoning can be applied to the representing tuple $(u, -f)$ with $-f$ strictly increasing.
Moreover part: "⇒": The statement will be proven in the respective moreover part of proposition 8. Replace g by f, T by N, $(x^*, ..., x^*)$ by x^* and $(x_1, x_2, ..., x_2)$ by $\frac{1}{N}x_i + \frac{N-1}{N}x_2$ to read the respective proof in terms of proposition 7. Note that the constructive proof for uniqueness up to affine transformations in an expected utility representation as given e.g. in Fishburn (1970, 114 et seq.) or Rubinstein (2006, 93 et seq.), which could be adapted here, would not carry over to the uniqueness of g in the intertemporal aggregation rule in proposition 8 because the latter applies fixed weights to every period.
"⇐": Follows immediately from $\mathcal{M}^f = \mathcal{M}^{\mathfrak{a}f}$ with $\mathfrak{a}f$ being a nondegenerate affine transformation of f (see beginning of Appendix B.1). □

[2] To see that $u^0(B)$ is indeed the preimage of A under s note that $s \circ u^0(B) = s \circ u^0(u^{-1}(A)) = u(u^{-1}(A)) = A$.

Proof of corollary 1: "⇒": By axioms A1-A3 and theorem 1 there exists a representation (u^0, id) of $\succeq$ in the sense of proposition 7 where id is the identity (defined on $U^0 \equiv \mathrm{range}(u^0)$). Then by lemma 0 also $(f^{-1}u^0, f)$ represents $\succeq$ in the sense of proposition 7 (and equation 6.2). Due to continuity of f^{-1} (see footnote 1) and u^0, the function $u \equiv f^{-1}u^0$ is continuous and constitutes a Bernoulli utility function for which $\mathcal{M}^f$ represents $\succeq$.
Remark with respect to footnote 5 on page 75: In the above construction equation (6.2) applies f only on the domain $f|_U : U \to U^0$. One can define f right away on a given, nondegenerate interval $U = [\underline{U}, \overline{U}] \subset \mathbb{R}$ and require the representing Bernoulli utility function $u : X \to U$ to be surjective. For this case define the affine transformation a^+ by $a = \frac{f(\underline{U}) - f(\overline{U})}{\underline{U}^0 - \overline{U}^0}$ and $b = \frac{f(\overline{U})\underline{U}^0 - f(\underline{U})\overline{U}^0}{\underline{U}^0 - \overline{U}^0}$. Then $u \equiv f^{-1}\mathfrak{a}^0 u^0$ renders the unique Bernoulli utility function satisfying equation (6.2) together with the above properties.
"⇐": As u in equation (6.2) is a Bernoulli utility function, this part of the proof is implied by the "⇐" part of proposition 7.
Moreover part: "⇒": Equation (6.2) implies for degenerate lotteries that there exists a strictly increasing function s such that $u' = s \circ u$. As in the proof of proposition 7 it follows that s is continuous. If $(u', f) = (s \circ u, f)$ is a representation of $\succeq$ then so is $(s^{-1} \circ s \circ u, f \circ s) = (u, f \circ s)$ by lemma 0. Comparing the latter with the representation (u, f) the moreover part of proposition 7 brings about the existence of $\mathfrak{a} \in \mathsf{A}$ such that $f = \mathfrak{a} f s$. From the fact that s is strictly increasing it can be inferred that $\mathfrak{a}$ is strictly increasing as well and can be replaced by $\mathfrak{a}^+ \in \mathsf{A}^+$. But then it follows $fu = \mathfrak{a}^+ f s u \Rightarrow fu = \mathfrak{a}^+ f u' \Rightarrow u = f^{-1}\mathfrak{a}^+ f u'$.
"⇐": First let f be increasing. If (u, f) is a representation of $\succeq$ then by proposition 7 also $(u, \mathfrak{a}^+ f)$ is a representation. By lemma 0 it follows that also $([\mathfrak{a}^+ f]u, \mathfrak{a}^+ f[\mathfrak{a}^+ f]^{-1})$ is a representation. Applying lemma 0 once again yields the result that $(f^{-1}\mathfrak{a}^+ f u, f)$ is a representation of $\succeq$. For f decreasing by proposition 7 $(u, -\mathfrak{a}^+ f)$ is a representation of $\succeq$ and by a similar reasoning as above so are $([-\mathfrak{a}^+ f]u, \mathrm{id})$, $(f^{-1}\{-[-\mathfrak{a}^+ f]u\}, -f)$, $(f^{-1}\mathfrak{a}^+ f u, -f)$ and $(f^{-1}\mathfrak{a}^+ f u, f)$. □

Proof of proposition 8: "⇒": By axiom A4 it exists u^0 such that $\sum_{i=1}^T u^0(x_i)$ represents $\succeq$. Furthermore as $u \in B_{\succeq}$ find that

$$\begin{array}{cccc} & u(x) & \geq & u(x') \\ \Leftrightarrow & [x]_1 & \succeq & [x']_1 \\ \Leftrightarrow & (x, x^0, ..., x^0) & \succeq & (x', x^0, ..., x^0) \\ \Leftrightarrow & u^0(x^1) + \sum_{i=2}^T u^0(x^0) & \geq & u^0(x^2) + \sum_{i=2}^T u^0(x^0) \end{array}$$

$$\Leftrightarrow u^0(x) \geq u^0(x').$$

Hence there exists a strictly increasing and continuous[3] transformation function $g : U \to \mathbb{R}$ such that $u^0 = g \circ u$. But then it is

$$\begin{aligned} \sum_{t=1}^{T} u^0(x_t) &\geq \sum_{t=1}^{T} u^0(x'_t) \\ \Leftrightarrow \quad \sum_{t=1}^{T} g \circ u(x_t) &\geq \sum_{t=1}^{T} g \circ u(x'_t) \\ \Leftrightarrow \quad \tfrac{1}{T}\sum_{t=1}^{T} g \circ u(x_t) &\geq \tfrac{1}{T}\sum_{t=1}^{T} g \circ u(x'_t) \\ \Leftrightarrow g^{-1}\left[\tfrac{1}{T}\sum_{t=1}^{T} g \circ u(x_t)\right] &\geq g^{-1}\left[\tfrac{1}{T}\sum_{t=1}^{T} g \circ u(x'_t)\right] \end{aligned}$$

yielding the representation stated in equation (6.4).
"$\Leftarrow$": For g increasing $u^0 \equiv g \circ u$ and for g decreasing $u^0 \equiv -g \circ u$ satisfy axiom A4.
Moreover part: "$\Rightarrow$": Given the Bernoulli utility function of proposition 8 I define $\overline{U} = \max_{x \in X} u(x)$ and $\underline{U} = \min_{x \in X} u(X)$ (the extrema are attained by continuity of u and compactness of X). Let $g' : U \to \mathbb{R}$ be another strictly monotonic, continuous function satisfying equation (6.4). If indifference between all outcomes holds (i.e. $\overline{U} = \underline{U}$), both functions g and g' have a degenerate codomain and they are trivially affine transformations of each other. Hence in the following it is assumed that $\overline{U} > \underline{U}$.

Define $\tilde{g} = ag' + b$ as the affine transformation of g' that coincides with g on the best and the worst outcome, i.e. $\tilde{g}(\overline{U}) \stackrel{!}{=} g(\overline{U})$ and $\tilde{g}(\underline{U}) \stackrel{!}{=} g(\underline{U})$ (corresponding to $a = \frac{g(\overline{U}) - g(\underline{U})}{g'(\overline{U}) - g'(\underline{U})}$ and $b = \frac{g'(\overline{U})g(\underline{U}) - g'(\underline{U})g(\overline{U})}{g'(\overline{U}) - g'(\underline{U})}$). In the following I will show by contradiction that g and $\tilde{g}$ have to coincide everywhere. The latter will imply that g' is in fact an affine transformation of g.

Assume that $\tilde{g}(u)$ and $g(u)$ do not coincide for all $u \in U$.[4] Then by continuity and connectedness there exist x_1 and x_2 with

$$u_1 \equiv u(x_1) < u_2 \equiv u(x_2) \tag{B.3}$$

such that $\tilde{g}(u_1) = g(u_1)$, $\tilde{g}(u_2) = g(u_2)$ and

$$\tilde{g}(u) \neq g(u) \ \ \forall\, u \in (u_1, u_2). \tag{B.4}$$

Let $(q_i)_{i \in \{1,\dots,T\}}$ denote a sequence of weights with $0 < q_i < 1 \ \forall i$ and $\sum_{i=1}^{T} q_i = 1$.

[3]Compare continuity of s in the proof of proposition 7.

[4]For the rest of this proof u also specifies particular values that the utility function u takes on. It should be obvious where u specifies a function and where it specifies a value.

The particular case of proposition 8 is $q_i = \frac{1}{T}\ \forall i$. By construction it is $q_1\tilde{g}(u_1) + \sum_{i=2}^{T} q_i\tilde{g}(u_2) = q_1 g(u_1) + \sum_{i=2}^{T} q_i g(u_2)$. Now equation (B.3) and continuity imply the existence of x^* with $u^* \equiv u(x^*)$ such that

$$u_1 < u^* < u_2 \tag{B.5}$$

and $(x^*, x^*, \ldots, x^*) \sim (x_1, x_2, \ldots, x_2)$. Therefore the following equivalence holds:

$$\begin{aligned}\tilde{g}(u^*) &= q_1\tilde{g}(u^*) + \sum_{i=2}^{T} q_i\tilde{g}(u^*) = q_1\tilde{g}(u_1) + \sum_{i=2}^{T} q_i\tilde{g}(u_2)\\ &= q_1 g(u_1) + \sum_{i=2}^{T} q_i g(u_2) = q_1 g(u^*) + \sum_{i=2}^{T} q_i g(u^*) = g(u^*).\end{aligned}$$

But $\tilde{g}(u^*) = g(u^*)$ contradicts equations (B.4) and (B.5). Hence $\tilde{g}$ and g have to coincide and g' is an affine transformation of g.

"$\Leftarrow$" is a special case of "$\Leftarrow$" in the moreover part of proposition 7. □

Proof of theorem 2: "$\Rightarrow$": First I show that by certainty additivity A4 and time consistency A5 it is $u \in B_{\succeq_F}$ *if and only if* $u \in B_{\succeq_T}$. Hereto note that by certainty additivity it exists $u^0 : X \to R$ such that

$$\begin{aligned} (x^0, x) &\succeq_F (x^0, x')\\ \Leftrightarrow u^0(x^0) + u^0(x) &\geq u^0(x^0) + u^0(x')\\ \Leftrightarrow u^0(x) + u^0(x^0) &\geq u^0(x') + u^0(x^0)\\ \Leftrightarrow (x, x^0) &\succeq_F (x', x^0).\end{aligned}$$

The statement $u \in B_{\succeq_T}$ is equivalent to $u(x) \geq u(x') \Leftrightarrow x \succeq_T x'$ which by time consistency and the above is equivalent to $u(x) \geq u(x') \Leftrightarrow (x^0, x) \succeq_F (x^0, x') \Leftrightarrow (x, x^0) \succeq_F (x', x^0)$. But the latter is the definition of $u \in B_{\succeq_F}$.

Second by A1-A3 *proposition 7 gives the existence of* f as in the proposition such that $\mathcal{M}^f$ represents $\succeq_T$. For any $p \in P$ define x^p to be an arbitrary element of the set of *certainty equivalents* $\{x \in X : u(x^p) = \mathcal{M}^f(p, u)\}$ for p. In the rest of this paragraph I show that these sets are not empty. As X is connected compact and u is continuous the range is a closed interval $u(X) = [\underline{U}, \overline{U}]$. Moreover $\max_p \mathcal{M}^f(p, u) = \max_x \mathcal{M}^f(\delta_x, u) = \max_x u(x) = \overline{U}$ and $\min_p \mathcal{M}^f(p, u) = \min_x \mathcal{M}^f(\delta_x, u) = \min_x u(x) = \underline{U}$. Hence $u^{-1}\left(\mathcal{M}^f(p, u)\right)$ is nonempty for all $p \in P$.

Third by definition and time consistency A5 it holds that $x^p \sim_T p \Leftrightarrow (x, x^p) \sim_F (x, p)\ \forall x \in X, p \in P$. From certainty additivity A4 and $u \in B_{\succeq}$ *proposition 8 brings about the existence of* g as in the proposition such that $(x, x^p) \succeq_F (x', x^{p'}) \Leftrightarrow g^{-1}\left[\frac{1}{2} g \circ u(x) + \frac{1}{2} g \circ u(x^p)\right] \geq g^{-1}\left[\frac{1}{2} g \circ u(x') + \frac{1}{2} g \circ u(x^{p'})\right]$.

Forth *combining* the above statements I find for any two pairs $(x,p),(x',p') \in X \times P$ that the following relation holds:

$$\begin{array}{cccc} & (x,p) & \succeq_F & (x',p') \\ \Leftrightarrow & (x,x^p) & \succeq_F & (x',x^{p'}) \\ \Leftrightarrow & g^{-1}\left[\frac{1}{2}g\circ u(x)+\frac{1}{2}g\circ u(x^p)\right] & \geq & g^{-1}\left[\frac{1}{2}g\circ u(x')+\frac{1}{2}g\circ u(x^{p'})\right] \\ \Leftrightarrow & g^{-1}\left[\frac{1}{2}g\circ u(x)+\frac{1}{2}g\circ \mathcal{M}^f(p,u)\right] & \geq & g^{-1}\left[\frac{1}{2}g\circ u(x')+\frac{1}{2}g\circ \mathcal{M}^f(p',u)\right]. \end{array}$$

This implies representation v).
"$\Leftarrow$": i) follows immediately from proposition 7. The same is true for ii) noting that $\mathcal{M}^f(\delta_x,u)=u(x)$ and using proposition 8. To show iii) first replace a decreasing g by $-g$ yielding the same representation v) with an increasing g. Then iii) can be seen to hold by $(x,p)\succeq_F(x,p') \Leftrightarrow g^{-1}\left[\frac{1}{2}g\circ u(x)+\frac{1}{2}g\circ\mathcal{M}^f(p,u)\right] \geq g^{-1}[\frac{1}{2}g\circ u(x)+\frac{1}{2}g\circ\mathcal{M}^f(p',u)] \Leftrightarrow$ $g\circ u(x)+g\circ\mathcal{M}^f(p,u) \geq g\circ u(x)+g\circ\mathcal{M}^f(p',u) \Leftrightarrow \mathcal{M}^f(p,u)\geq\mathcal{M}^f(p',u) \Leftrightarrow p\succeq_T p'$.
Moreover part: For g the moreover part is immediate from proposition 8. For f note that affine transformations of f leave $\mathcal{M}^f$ unchanged, so that the affine transformations allowed for f in vi) by proposition 7 also leave the representative character of v) untouched. □

Proof of lemma 1: The triple $(s\circ u, f\circ s^{-1}, g\circ s^{-1})$ stands for the representation $sg^{-1}\left[\frac{1}{2}gs^{-1}su(x)+\frac{1}{2}gs^{-1}sf^{-1}\left[\int_X(fs^{-1})(su)\,dp\right]\right] = sg^{-1}\left[\frac{1}{2}gu(x)+\frac{1}{2}gf^{-1}\left[\int_X(fu)\,dp\right]\right]$. The latter is a strictly increasing transformation of the representation of $\succeq_F$ corresponding to the triple (u,f,g). Moreover $s\circ u$, $f\circ s^{-1}$ and $g\circ s^{-1}$ are continuous (see footnote 1) and $f\circ s^{-1}$ and $g\circ s^{-1}$ are strictly monotonic. Hence $(s\circ u, f\circ s^{-1}, g\circ s^{-1})$ is a representation for $\succeq_F$ itself. The same reasoning obviously holds for $\succeq_T$. □

Proof of corollary 2: "$\Rightarrow$": By certainty additivity A4 the set of Bernoulli utility functions $B_{\succeq}=B_{\succeq_T}$ is not empty. Hence by theorem 2 there exists a representing tuple (u^0,f^0,g^0) for some $u^0\in B_{\succeq}$. Wlog let f^0 be increasing if and only if f given in the corollary is increasing (if this is not the case just take $-f^0$ rendering a representation for the same preference). Then $s=f^{-1}f^0$ is strictly increasing and continuous (see footnote 1). Lemma 1 zields that $([f^{-1}f^0]u^0, f^0[f^{-1}f^0]^{-1}, g^0[f^{-1}f^0]^{-1}) = (f^{-1}f^0u^0, f, g^0f^{0^{-1}}f)$ is a representation of $\succeq$ that uses f to characterize the the uncertainty aggregation rule.
"$\Leftarrow$": Implied by "$\Leftarrow$" in theorem 2.
Moreover part: "$\Rightarrow$": Let (u,f,g) and (u',f,g') be representations for $\succeq$ in the sense of v) and vi). Then looking at degenerate lotteries vi) implies that there exists a strictly

increasing continuous transformation such that $u' = s \circ u$. If $(u', f, g') = (s\,u, f, g')$ is a representation of $\succeq$ then by lemma 1 so is the triple $(u, fs, g's)$. Comparing the latter to the representation (u, f, g) the moreover part of theorem 2 implies that there have to exist affine transformations $\mathfrak{a}^+ \in \mathsf{A}^+$ and $\mathfrak{a} \in \mathsf{A}$ such that

$$f = \mathfrak{a}^+ f\,s \quad \Leftrightarrow \quad s^{-1} = f^{-1} a^+ f \quad \text{and}$$
$$g = \mathfrak{a} g' s.$$

That $\mathfrak{a}^+$ has to be *positive* affine can be inferred from the fact that s is strictly increasing. Substituting the result for s into the equations for g and u renders

$$g = \mathfrak{a} g' f^{-1} a^{+^{-1}} f \quad \text{and}$$
$$u = s^{-1} u' = f^{-1} a^+ f\, u'.$$

"$\Leftarrow$": First let f be increasing. If (u, f, g) is a representation of $\succeq$ then by theorem 2 for $\mathfrak{a}^+ \in \mathsf{A}^+$ and $\mathfrak{a} \in \mathsf{A}$ also $(u, \mathfrak{a}^+ f, \mathfrak{a} g)$ is a representation. By lemma 1 it follows that also $([\mathfrak{a}^+ f]u, \mathfrak{a}^+ f[\mathfrak{a}^+ f]^{-1}, \mathfrak{a} g[\mathfrak{a}^+ f]^{-1}) = (\mathfrak{a}^+ f u, \mathrm{id}, \mathfrak{a} g f^{-1} \mathfrak{a}^{+^{-1}})$ is a representation. Applying lemma 1 once again yields the result that $(f^{-1}\mathfrak{a}^+ f u, f, \mathfrak{a} g f^{-1} \mathfrak{a}^{+^{-1}} f)$ is a representation of $\succeq$. For f decreasing find that by theorem 2 $(u, -\mathfrak{a}^+ f, \mathfrak{a} g)$ is a representation of $\succeq$ and by a similar reasoning as above so are $([-\mathfrak{a}^+ f]u, \mathrm{id}, \mathfrak{a} g f^{-1}(-\mathfrak{a}^+)^{-1})$, $(f^{-1}\{-[-\mathfrak{a}^+ f]u\}, -f, \mathfrak{a} g f^{-1}(-\mathfrak{a}^+)^{-1}(-f))$ and $(f^{-1}\mathfrak{a}^+ f u, -f, \mathfrak{a} g f^{-1}(\mathfrak{a}^+)^{-1} f)$ as well as $(f^{-1}\mathfrak{a}^+ f u, f, \mathfrak{a} g f^{-1} \mathfrak{a}^{+^{-1}} f)$. □

Proof of corollary 3: Imitates the proof of corollary 2. □

B.4 Proofs for Chapter 7

Proof of lemma 2: Let $u \in B_{\succeq}$ be an arbitrary Bernoulli utility function for the set of preference relations $\succeq$. By theorem 2 there exist f and g as in the theorem, unique up to nondegenerate affine transformations, such that the triple (u, f, g) represents $\succeq$. Let $\tilde{u} \in B_{\succeq}$ be any other Bernoulli utility function for $\succeq$ and let the triple $(\tilde{u}, \tilde{f}, \tilde{g})$ be a corresponding representation in the sense of theorem 2.

Compare proof of proposition 7 to see that there exists a strictly increasing continuous transformation s such that $\tilde{u} = s \circ u$. Lemma 1 implies that with (u, f, g) as well $(s \circ u, f \circ s^{-1}, g \circ s^{-1}) = (\tilde{u}, f \circ s^{-1}, g \circ s^{-1})$ is a representation for $\succeq$. Comparing the latter to the representation $(\tilde{u}, \tilde{f}, \tilde{g})$ the moreover part of theorem 2 implies the existence of affine transformations $\mathfrak{a}, \tilde{\mathfrak{a}} \in \mathsf{A}$ such that $\tilde{f} = \mathfrak{a} f \circ s^{-1}$ and $\tilde{g} = \tilde{\mathfrak{a}} g \circ s^{-1}$. But then find that $\tilde{f} \circ \tilde{g}^{-1} = \mathfrak{a} f \circ s^{-1} \circ s \circ g^{-1} \tilde{\mathfrak{a}}^{-1} = \mathfrak{a} f \circ g^{-1} \tilde{\mathfrak{a}}^{-1} \in \hat{f} \circ \hat{g}^{-1}$. □

Proof of theorem 3: a) "$\Rightarrow$": The first part of the premise in axiom A6$^{\mathrm{s}}_{\mathrm{nd}}$ translates into the representation of theorem 2 as

$$\begin{array}{rcl} (\bar{x},\bar{x}) & \sim_F & (x_1,x_2) \\ \Leftrightarrow g^{-1}\left(\tfrac{1}{2}g\circ u(\bar{x})+\tfrac{1}{2}g\circ u(\bar{x})\right) & = & g^{-1}\left(\tfrac{1}{2}g\circ u(x_1)+\tfrac{1}{2}g\circ u(x_2)\right) \\ \Leftrightarrow \qquad u(\bar{x}) & = & g^{-1}\left(\tfrac{1}{2}g\circ u(x_1)+\tfrac{1}{2}g\circ u(x_2)\right) \end{array} \tag{B.6}$$

and for the second part of the premise renders

$$\begin{array}{rcl} x_1 & \not\sim_T & x_2 \\ \Leftrightarrow u(x_1) & \neq & u(x_2)\,. \end{array} \tag{B.7}$$

Writing the implication of axiom A6$^{\mathrm{s}}_{\mathrm{nd}}$ in the representation of theorem 2 yields

$$\begin{array}{rcl} \bar{x} & \succ_T & \tfrac{1}{2}x_1+\tfrac{1}{2}x_2 \\ \Leftrightarrow u(\bar{x}) & > & f^{-1}\left(\tfrac{1}{2}f\circ u(x_1)+\tfrac{1}{2}f\circ u(x_2)\right)\,. \end{array} \tag{B.8}$$

Combining equations (B.6) and (B.8) renders

$$g^{-1}\left(\tfrac{1}{2}g\circ u(x_1)+\tfrac{1}{2}g\circ u(x_2)\right) > f^{-1}\left(\tfrac{1}{2}f\circ u(x_1)+\tfrac{1}{2}f\circ u(x_2)\right) \tag{B.9}$$

which for an increasing [decreasing] version of f is equivalent to

$$\Leftrightarrow f\circ g^{-1}\left(\tfrac{1}{2}g\circ u(x_1)+\tfrac{1}{2}g\circ u(x_2)\right) > [<]\ \tfrac{1}{2}f\circ u(x_1)+\tfrac{1}{2}f\circ u(x_2)\,.$$

Defining $z_i = g\circ u(x_i)$ the equation becomes

$$\Leftrightarrow f\circ g^{-1}\left(\tfrac{1}{2}z_1+\tfrac{1}{2}z_2\right) > [<]\ \tfrac{1}{2}f\circ g^{-1}(z_1)+\tfrac{1}{2}f\circ g^{-1}(z_2)\,. \tag{B.10}$$

Noting that for all $x_1,x_2\in X$ there exists a certainty equivalent $\bar{x}$ to the lottery $\frac{1}{2}x_1+\frac{1}{2}x_2$ (compare proof of theorem 2) so that the first part of the premise is satisfied and that the second part of the premise implies (by equation B.7) that $z_1\neq z_2$, one finds that equation (B.10) has to hold for all $z_1,z_2\in\Gamma$ with $z_1\neq z_2$. Therefore $f\circ g^{-1}$ has to be strictly concave [convex] on Γ (Hardy et al. 1964, 75).

a) "$\Leftarrow$": The necessity of axiom A6$^{\mathrm{s}}_{\mathrm{nd}}$ is seen to hold by mainly going backwards through the proof of sufficiency given above. Strict concavity [convexity] of $f\circ g^{-1}$ with f increasing [decreasing] implies that equation (B.10) and thus equation (B.9) have to hold for $z_1,z_2\in\Gamma$ with $z_1\neq z_2$. The latter corresponds to the second part of the premise of axiom A6$^{\mathrm{s}}_{\mathrm{nd}}$ (see equation B.7). The second part of the premise corresponding to (B.6) guarantees that equation (B.9) implies equation (B.8) which yields the implication of axiom A6$^{\mathrm{s}}_{\mathrm{nd}}$.

b): Replace in the proof of part a) 'strict' by 'weak', '$\succ_T$' by '$\succeq_T$' and '$>$' by '$\geq$' and the proof is valid for part b).

c)"$\Rightarrow$": Replacing in the reasoning of "a $\Rightarrow$" the symbols '$\succ_T$' by '$\sim_T$' and '$>$' by '$=$'

yields the following alteration of equation (B.10)

$$f \circ g^{-1}\left(\tfrac{1}{2}z_1 + \tfrac{1}{2}z_2\right) = \tfrac{1}{2} f \circ g^{-1}(z_1) + \tfrac{1}{2} f \circ g^{-1}(z_2)$$

which has to hold for all $z_1, z_2 \in \Gamma$ with $z_1 \neq z_2$. By Aczél (1966, 46) this condition implies linearity of $f \circ g^{-1}$ on Γ. Hence $f \circ g^{-1} = \mathfrak{a}$ for some $\mathfrak{a} \in \mathsf{A}$. Then gauging $f = \text{id}$ yields $g = \mathfrak{a}^{-1}$. But as g is only determined up to affine transformations I can represent the same preference with choosing $g = \text{id}$ as well. However with $f = g = \text{id}$ the representation of theorem 2 represents $\succeq_F$ as intertemporally additive expected utility.
c)"$\Leftarrow$": If the preference relation $\succeq$ is representable by intertemporal expected utility, then the premise of axiom A6$^{\text{w}}_{\text{nd}}$, respectively the first part of the premise of axiom A6$^{\text{s}}_{\text{nd}}$, translates into

$$\begin{array}{rccc} & (\bar{x}, \bar{x}) & \sim_F & (x_1, x_2) \\ \Leftrightarrow & u(\bar{x}) + u(\bar{x}) & = & u(x_1) + u(x_2) \\ \Leftrightarrow & u(\bar{x}) & = & \frac{1}{2}\left(u(x_1) + u(x_2)\right) \\ \Leftrightarrow & \bar{x} & \sim_F & \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{array}$$

and thus implies intertemporal risk neutrality. □

Proof of proposition 9: Let h be some function in $\hat{f} \circ \hat{g}^{-1}$. Then any other function $\tilde{h}$ in $\hat{f} \circ \hat{g}^{-1}$ can be expressed as $\tilde{h} = \mathfrak{a} h \tilde{\mathfrak{a}}^{-1}$ for some $\mathfrak{a}, \tilde{\mathfrak{a}} \in \mathsf{A}$. Strict concavity of h is equivalent to

$$h\left(\tfrac{1}{2}z_1 + \tfrac{1}{2}z_2\right) > \tfrac{1}{2}h(z_1) + \tfrac{1}{2}h(z_2)$$

for all $z_1, z_2 \in \Gamma$ with $z_1 \neq z_2$ (Hardy et al. 1964, 75). Furthermore with introducing $\tilde{z}_i = \tilde{\mathfrak{a}} z_i, i \in 1, 2$, the following manipulations hold

$$\begin{array}{rccc} \Leftrightarrow & \mathfrak{a}h\left(\frac{1}{2}z_1 + \mathfrak{a}\frac{1}{2}z_2\right) & > & \frac{1}{2}\mathfrak{a}h(z_1) + \frac{1}{2}\mathfrak{a}h(z_2) \\ \Leftrightarrow & \mathfrak{a}h\left(\frac{1}{2}\tilde{\mathfrak{a}}^{-1}\tilde{z}_1 + \frac{1}{2}\tilde{\mathfrak{a}}^{-1}\tilde{z}_2\right) & > & \frac{1}{2}\mathfrak{a}h\left(\tilde{\mathfrak{a}}^{-1}z_1\right) + \frac{1}{2}\mathfrak{a}h\left(\tilde{\mathfrak{a}}^{-1}z_2\right) \\ \Leftrightarrow & \mathfrak{a}h\tilde{\mathfrak{a}}^{-1}\left(\frac{1}{2}\tilde{z}_1 + \frac{1}{2}\tilde{z}_2\right) & > & \frac{1}{2}\mathfrak{a}h\tilde{\mathfrak{a}}^{-1}(z_1) + \frac{1}{2}\mathfrak{a}h\tilde{\mathfrak{a}}^{-1}(z_2) \\ \Leftrightarrow & \tilde{h}\left(\frac{1}{2}\tilde{z}_1 + \frac{1}{2}\tilde{z}_2\right) & > & \frac{1}{2}\tilde{h}(\tilde{z}_1) + \frac{1}{2}\tilde{h}(\tilde{z}_2) \end{array}$$

for all $\tilde{z}_1, \tilde{z}_2 \in \tilde{\Gamma} = \tilde{\mathfrak{a}}\Gamma$ with $\tilde{z}_1 \neq \tilde{z}_2$. But the last equation characterizes strict concavity of $\tilde{h}$. The same reasoning holds true for weak concavity replacing $>$ for $\geq$ and for convexity reversing the inequalities. □

Proof of corollary 4: By corollary 3 there exists a Bernoulli utility function u and a function f such that in a $g = \bar{g}$-gauge the triple $(u, f, \bar{g})$ represents the set of preference relations $\succeq$. The allowed transformations for u and f without fixing the range of u are $(u, f) \rightarrow (\tilde{u}, \tilde{f}) = (g^{-1}\mathfrak{a}^{+^{-1}}g\ u, \mathfrak{a}\ f\ g^{-1}\mathfrak{a}^{+}g)$ with $\mathfrak{a} \in \mathsf{A}$ and $\mathfrak{a}^{+} \in \mathsf{A}^{+}$. Now write the closed interval W^* as $[\underline{U},\overline{U}]$ and require that some representing Bernoulli utility function $\tilde{u}$ satisfies $\tilde{u}(x_{\min}) = \underline{U}$ and $\tilde{u}(x_{\max}) = \overline{U}$, where $x_{\min}$ and $x_{\max}$ are elements of the set of worst respectively best outcomes in X (which is non-empty due to compactness and continuity). Starting from the arbitrary Bernoulli utility function u an appropriate $\tilde{u}$ is obtained as $\tilde{u} = \bar{g}^{-1}\mathfrak{a}^{+^{-1}}\bar{g}\,u$ by means of the affine transformation $\mathfrak{a}^+$ satisfying the conditions $\mathfrak{a}^+\left(\bar{g}\circ u(x_{\min})\right) \stackrel{!}{=} \bar{g}(\underline{U})$ and $\mathfrak{a}^+\left(\bar{g}\circ u(x_{\max})\right) \stackrel{!}{=} \bar{g}(\overline{U})$. Being an affine function, $\mathfrak{a}^+$ is determined uniquely by fixing two of its points. Thus, $\tilde{u}$ is determined uniquely and the freedom remaining for the representing function f reduces to $f \to \tilde{f} = \mathfrak{a}f$ with $\mathfrak{a} \in \mathsf{A}$. Therefore, the expression for intertemporal risk aversion $f\circ\bar{g}^{-1}$ is determined up to transformations $f\circ\bar{g}^{-1} \to \tilde{f}\bar{g}^{-1} = \mathfrak{a}f\bar{g}^{-1}$, rendering the according measures unique. □

Proof of corollary 5: This is a trivial consequence of theorem 2. Everything but the uniqueness of g has been established there. But fixing two points of a function g that is determined already up to affine transformations obviously removes the indeterminacy completely. □

Proof of lemma 3: Let the triples (u,f,g) and $(\tilde{u},\tilde{f},\tilde{g})$ be arbitrary representations for the set of preference relations $\succeq = (\succeq_F, \succeq_T)$. As in the proof of lemma 2 it follows that there exists a strictly increasing continuous function s and affine transformations $\mathfrak{a},\tilde{\mathfrak{a}} \in \mathsf{A}$ such that $\tilde{u} = s\circ u$, $\tilde{f} = \mathfrak{a}f\circ s^{-1}$ and $\tilde{g} = \tilde{\mathfrak{a}}g\circ s^{-1}$. Let $x^{\min}$ and $x^{\max}$ be (members of the set of) worst respectively best outcomes in X with respect to $\succeq$. Writing the closed interval G as $G = [\underline{G},\overline{G}]$ the additional assumption in lemma 3 requires for the function g that $\underline{G} = g\circ u(x^{\min})$ and $\overline{G} = g\circ u(x^{\max})$. Similarly for $\tilde{g}$ it has to hold that $\underline{G} = \tilde{g}\circ\tilde{u}(x^{\min})$ and $\overline{G} = \tilde{g}\circ\tilde{u}(x^{\max})$. Together with the implication of lemma 2 these requirements bring about the relations

$$\underline{G} = \tilde{g}\circ\tilde{u}(x^{\min}) = \tilde{\mathfrak{a}}g\circ s^{-1}\circ\tilde{u}(x^{\min}) = \tilde{\mathfrak{a}}g\circ u(x^{\min}) = \tilde{\mathfrak{a}}\,\underline{G} \quad \text{and}$$
$$\overline{G} = \tilde{g}\circ\tilde{u}(x^{\max}) = \tilde{\mathfrak{a}}g\circ s^{-1}\circ\tilde{u}(x^{\max}) = \tilde{\mathfrak{a}}g\circ u(x^{\max}) = \tilde{\mathfrak{a}}\,\overline{G}\,.$$

Therefore $\tilde{\mathfrak{a}}$ has to be the identity and it is $\tilde{f}\circ\tilde{g}^{-1} = \mathfrak{a}f\circ s^{-1}\circ s\circ g^{-1}\tilde{\mathfrak{a}}^{-1} = \mathfrak{a}f\circ g^{-1}$. But by construction the measures of intertemporal risk aversion RIRA and AIRA are independent of the remaining indeterminacy corresponding to the transformation $\mathfrak{a}$. □

Appendix C

Proofs for Part III

C.1 Proofs for Chapter 8

Proof of theorem 4: The proof is divided into four parts. The first part gives a representation for certain consumption paths which is a generalization of proposition 8 for non-stationary utility. In part two I derive a corresponding recursive formulation, still only for certain consumption paths. Finally, part three works out the general representation for temporal lotteries as given in the theorem. Part four verifies that the derived representation implies all axioms.

Part I ("⇒"): First, note that certainty additivity A4 for $\succeq_1$ carries over to $\succeq_t$ for all t with *coinciding* Bernoulli utility functions $u^o_{\tau,\tau\geq t}$. The argument works inductively. Given that $\succeq_t|_{\mathsf{X}_1}$ has a certainty additive representation with Bernoulli utility functions $u^o_{\tau,\tau\geq t}$, it follows from time consistency A5 that for all $\mathsf{x}^{t+1}, \mathsf{x}'^{t+1} \in \mathsf{X}^{t+1}$ and any $x_t \in X_t$:

$$\begin{array}{lccc} & \mathsf{x}^{t+1} & \succeq_{t+1} & \mathsf{x}'^{t+1} \\ \Leftrightarrow & (x_t, \mathsf{x}^{t+1}) & \succeq_t & (x_t, \mathsf{x}'^{t+1}) \\ \Leftrightarrow & u^o_t(x_t) + \sum_{\tau=t+1}^T u^o_\tau(\mathsf{x}^{t+1}_\tau) & \geq & u^o_t(x_t) + \sum_{\tau=t+1}^T u^o_\tau(\mathsf{x}'^{t+1}_\tau) \\ \Leftrightarrow & \sum_{\tau=t+1}^T u^o_\tau(\mathsf{x}^{t+1}_\tau) & \geq & \sum_{\tau=t+1}^T u^o_\tau(\mathsf{x}'^{t+1}_\tau). \end{array}$$

Therefore $\succeq_{t+1}$ has a certainty additive representation which uses the same Bernoulli utility functions u^o_τ for $\tau \geq t+1$ as does the above representation for $\succeq_t$. In the following u^o_t continues to denote the above utility index derived from certainty additivity, while u_t denotes the period t (Bernoulli-) utility function given in the theorem.

Second, I show that for every pair of utility functions u^o_t and u_t there exists a strictly increasing, continuous transformation g_t such that $u_t = g_t \circ u^o_t$. By $u_\tau \in B_{\succeq_t}$ I have:

$$\begin{array}{lccc} & u_t(x_t) & \geq & u_t(x'_t) \\ \Leftrightarrow & [x_t]_t & \succeq_t & [x'_t]_t \\ \Leftrightarrow & (x_t, x^o, ..., x^o) & \succeq_t & (x'_t, x^o, ..., x^o) \\ \Leftrightarrow & u^o_t(x_t) + \sum_{\tau=t}^T u^o_\tau(x_\tau) & \geq & u^o_t(x'_t) + \sum_{\tau=t}^T u^o_\tau(x_\tau) \\ \Leftrightarrow & u^o_t(x_t) & \geq & u^o_t(x'_t) \end{array}$$

Hence u_t is a monotonic transformation of u^o_t and it exists a strictly increasing function $g_t : U_t \rightarrow \mathbb{R}$ such that $u^o_t = g_t \circ u_t$. For the fact that continuity of u^o_t and u_t imply continuity of g_t consult the proof of proposition 7.

Third, I give a representation over certain consumption paths in terms of the Bernoulli utility functions $u_t, t \in \{1, ..., T\}$, given in the theorem. This is merely a task of combining the two results derived above which yield for all t and all $\mathsf{x}^t, \mathsf{x}'^t \in \mathsf{X}^t$:

$$\begin{array}{lccc} & \mathsf{x}^t & \succeq_t & \mathsf{x}'^t \\ \Leftrightarrow & \sum_{\tau=t}^T u^o_\tau(\mathsf{x}^t_\tau) & \geq & \sum_{\tau=t}^T u^o_\tau(\mathsf{x}'^t_\tau) \\ \Leftrightarrow & \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}^t_\tau) & \geq & \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}'^t_\tau). \end{array}$$

Part II ("⇒"): In this part, I construct the recursive analogue to the above representation for certain consumption paths. Let me first define for every $t \in \{1, ..., T-1\}$ the intertemporal aggregation rule

$$\begin{array}{ll} \mathcal{N}^{g_t, g_{t+1}} & : \; U_t \times U_{t+1} \rightarrow \mathbb{R} \\ \mathcal{N}^{g_t, g_{t+1}}(\cdot, \cdot) & = g_t^{-1} \left[\theta_t g_t(\cdot) + \theta_t \theta_{t+1}^{-1} g_{t+1}(\cdot) + \theta_t \theta_{t+1}^{-1} \vartheta_t \right] \end{array}$$

with normalization constants

$$\theta_t = \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau} \quad \text{and} \quad \vartheta_t = \frac{\overline{G}_{t+1} \underline{G}_t - \underline{G}_{t+1} \overline{G}_t}{\Delta G_t}.$$

The normalization constants ensure that the domain of g_t^{-1} in the intertemporal aggregation rule is in fact $[\underline{G}_t, \overline{G}_t]$. This will be verified in the rest of this paragraph. To this purpose, note that

$$\begin{array}{l} \overline{G}_{t+1} + \vartheta_t = \frac{\overline{G}_{t+1}(\overline{G}_t - \underline{G}_t) + \overline{G}_{t+1} \underline{G}_t - \underline{G}_{t+1} \overline{G}_t}{\Delta G_t} = \frac{\Delta G_{t+1}}{\Delta G_t} \overline{G}_t \quad \text{and} \\ \underline{G}_{t+1} + \vartheta_t = \frac{\underline{G}_{t+1}(\overline{G}_t - \underline{G}_t) + \overline{G}_{t+1} \underline{G}_t - \underline{G}_{t+1} \overline{G}_t}{\Delta G_t} = \frac{\Delta G_{t+1}}{\Delta G_t} \underline{G}_t \, . \end{array}$$

The maximal value of the argument of $g_t^{-1}[\,\cdot\,]$ in $\mathcal{N}^{g_t, g_{t+1}}$ is taken on for $\overline{G}_t = g_t(\overline{U}_t)$

and $\overline{G}_{t+1} = g_{t+1}(\overline{U}_{t+1})$ which yields

$$\begin{aligned}\theta_t\left[g_t(\cdot)+\theta_{t+1}^{-1}\left\{g_{t+1}(\cdot)+\vartheta_t\right\}\right]^{\max} &= \tfrac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\left[\overline{G}_t + \tfrac{\sum_{\tau=t+1}^T \Delta G_\tau}{\Delta G_{t+1}}\left\{\overline{G}_{t+1}+\vartheta_t\right\}\right]\\ &= \tfrac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\left[\overline{G}_t + \tfrac{\sum_{\tau=t+1}^T \Delta G_\tau}{\Delta G_{t+1}}\left\{\tfrac{\Delta G_{t+1}}{\Delta G_t}\,\overline{G}_t\right\}\right]\\ &= \tfrac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\left[\tfrac{\overline{G}_t\Delta G_t+\overline{G}_t\sum_{\tau=t+1}^T\Delta G_\tau}{\Delta G_t}\right] = \overline{G}_t .\end{aligned}$$

The minimal value of the argument of $g_t^{-1}[\,\cdot\,]$ in $\mathcal{N}^{g_t,g_{t+1}}$ is taken on for $\underline{G}_t = g_t(\underline{U}_t)$ and $\underline{G}_{t+1} = g_{t+1}(\underline{U}_{t+1})$. In this case exactly the same equation holds true with $\overline{G}_t$ replaced by $\underline{G}_t$. Hence the expression defining the intertemporal aggregation rule $\mathcal{N}^{g_t,g_{t+1}}$ is well defined.

For the second step, I introduce the notation ${}^t\mathrm{x}^{t-1}$ to denote the continuation of the consumption path $\mathrm{x}^{t-1}\in\mathrm{X}^{t-1}$ from period t on, i.e. $\mathrm{x}^{t-1} = (\mathrm{x}_{t-1}^{t-1}, {}^t\mathrm{x}^{t-1})$. Then, define the aggregate intertemporal utility functions *for certain consumptions paths* by setting $\tilde{u}_T = u_T$ and for $1 < t \le T$ recursively:

$$\begin{aligned}\tilde{u}_{t-1}(\mathrm{x}^{t-1}) \equiv \tilde{u}_{t-1}(\mathrm{x}_{t-1}^{t-1},\, {}^t\mathrm{x}^{t-1}) &= \mathcal{N}_*^{g_{t-1},g_t}\left(u_{t-1}(\mathrm{x}_{t-1}^{t-1}), \tilde{u}_t({}^t\mathrm{x}^{t-1})\right)\\ &= g_{t-1}^{-1}\left[\theta_{t-1}\, g_{t-1}\circ u_{t-1}(\mathrm{x}_{t-1}^{t-1}) + \theta_{t-1}\theta_t^{-1}\, g_t\circ\tilde{u}_t({}^t\mathrm{x}^{t-1}) + \theta_{t-1}\theta_t^{-1}\vartheta_{t-1}\right].\end{aligned}$$

From the first step in this part it follows that range$(\tilde{u}_t) = [\underline{U}_t, \overline{U}_t]$.

Third, I show that there exist constants ξ_t, such that the following equation holds for all $t\in\{1,..,T\}$:

$$\theta_t^{-1}\, g_t\circ\tilde{u}_t(\mathrm{x}^t) = \textstyle\sum_{\tau=t}^T g_\tau\circ u_\tau(\mathrm{x}_\tau^t) + \xi_t\,. \tag{C.1}$$

As $\theta_T = 1$ this relation obviously holds for $t = T$ (with $\xi_T = 0$). The following manipulation shows that the equation holds by (backwards) induction for all t:

$$\begin{aligned}&\theta_{t-1}^{-1}\, g_{t-1}\circ\tilde{u}_{t-1}(\mathrm{x}^{t-1})\\ &= \theta_{t-1}^{-1}\, g_{t-1}\circ g_{t-1}^{-1}\left[\theta_{t-1}\, g_{t-1}\circ u_{t-1}(\mathrm{x}_{t-1}^{t-1}) + \theta_{t-1}\theta_t^{-1}\, g_t\circ\tilde{u}_t({}^t\mathrm{x}^{t-1}) + \theta_{t-1}\theta_t^{-1}\vartheta_{t-1}\right]\\ &= g_{t-1}\circ u_{t-1}(\mathrm{x}_{t-1}^{t-1}) + \underbrace{\theta_t^{-1}\, g_t\circ\tilde{u}_t({}^t\mathrm{x}^{t-1})} + \theta_t^{-1}\vartheta_{t-1}\\ &= g_{t-1}\circ u_{t-1}(\mathrm{x}_{t-1}^{t-1}) + \textstyle\sum_{\tau=t}^T g_\tau\circ u_\tau(\mathrm{x}_\tau^{t-1}) + \underbrace{\xi_t + \theta_t^{-1}\vartheta_{t-1}}\\ &= \textstyle\sum_{\tau=t-1}^T g_\tau\circ u_\tau(\mathrm{x}_\tau^{t-1}) \qquad + \qquad \xi_{t-1}\,.\end{aligned}$$

But (C.1) states that on certain consumption paths $\tilde{u}_t$ is a (strictly) increasing transformation of $\sum_{\tau=t}^T g_\tau\circ u_\tau$ and hence *a representation of* $\succeq_t|_{\mathrm{X}^t}$.

Note: The following equality holds:

$$\begin{aligned}\theta_t\theta_{t+1}^{-1} &= \tfrac{\Delta G_t}{\sum_{\tau=t}^T\Delta G_\tau}\tfrac{\sum_{\tau=t+1}^T\Delta G_\tau}{\Delta G_{t+1}} = \tfrac{\Delta G_t}{\sum_{\tau=t}^T\Delta G_\tau}\tfrac{\sum_{\tau=t}^T\Delta G_\tau}{\Delta G_{t+1}} - \tfrac{\Delta G_t}{\sum_{\tau=t}^T\Delta G_\tau}\tfrac{\Delta G_t}{\Delta G_{t+1}}\\ &= (1-\theta_t)\tfrac{\Delta G_t}{\Delta G_{t+1}}.\end{aligned} \tag{C.2}$$

Part III ("⇒"): The extension of the representation to uncertainty employs backward induction. The induction hypothesis I have to proof is the following. For every $t \in \{1, ..., T\}$ and $\tilde{u}_t$ defined as in the theorem, it holds

H1 $\exists f_t : U_t \to \mathrm{I\!R}$ s.th. $p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t$.

The proof uses recursively the following additional hypothesis claiming that for every lottery there exists a certainty equivalent that is a certain consumption path:

H2 For all $p_t \in P_t$ there exists $\mathsf{x}^{p_t} \in \mathsf{X}^t$ such that $\mathsf{x}^{p_t} \sim p_t$.

Let me first verify that induction hypothesis H1 and H2 are satisfied for $t = T$. For H1 this is an immediate consequence of proposition 7. Observing that for $t = T$ it is $x_t = \mathsf{x}^t$, also H2 is an immediate consequence of the existence of a certainty equivalent for every $p_T \in P_T$ which has been shown in the proof of proposition 7.

Given H1 and H2 for t I proceed to show that the induction hypotheses also hold for $t-1$. To this end, note that $\mathcal{M}^{f_t}(p_t, \tilde{u}_t) = \mathcal{M}^{f_t}(\mathsf{x}^{p_t}, \tilde{u}_t) = \tilde{u}_t(\mathsf{x}^{p_t})$ and find that the following equivalence holds:

$$
\begin{array}{cccc}
 & (x_{t-1}, p_t) & \succeq_{t-1} & (x'_{t-1}, p'_t) \\
\Leftrightarrow & (x_{t-1}, \mathsf{x}^{p_t}) & \succeq_{t-1} & (x'_{t-1}, \mathsf{x}^{p'_t}) \\
\Leftrightarrow & \tilde{u}_{t-1}(x_{t-1}, \mathsf{x}^{p_t}) & \geq & \tilde{u}_{t-1}(x'_{t-1}, \mathsf{x}^{p'_t}) \\
\Leftrightarrow & \mathcal{N}^{g_{t-1}, g_t}\left(u_{t-1}(x_{t-1}), \tilde{u}_t(\mathsf{x}^{p_t})\right) & \geq & \mathcal{N}^{g_{t-1}, g_t}\left(u_{t-1}(x'_{t-1}), \tilde{u}_t(\mathsf{x}^{p'_t})\right) \\
\Leftrightarrow & \mathcal{N}^{g_{t-1}, g_t}\left(u_{t-1}(x_{t-1}), \mathcal{M}^{f_t}(p_t, \tilde{u}_t)\right) & \geq & \mathcal{N}^{g_{t-1}, g_t}\left(u_{t-1}(x'_{t-1}), \mathcal{M}^{f_t}(p'_t, \tilde{u}_t)\right) \\
\Leftrightarrow & \tilde{u}_{t-1}(x_{t-1}, p_t) & \geq & \tilde{u}_{t-1}(x'_{t-1}, p'_t)\,,
\end{array}
$$

where $\tilde{u}_{t-1}$ is the aggregate intertemporal utility function for degenerate period $t-1$ lotteries as given in the theorem. $\tilde{u}_{t-1} \in \mathcal{C}^0(X_{t-1} \times P_t)$ satisfies $(x_{t-1}, p_t) \succeq_{t-1} (x'_{t-1}, p'_t) \Leftrightarrow \tilde{u}_{t-1}(x_{t-1}, p_t) \geq \tilde{u}_{t-1}(x'_{t-1}, p'_t)$ for all $(x_{t-1}, p_t), (x'_{t-1}, p'_t) \in X_{t-1} \times P_t$. Therefore, by proposition 7 with the compact metric space $X_{t-1} \times P_t$, it exists $f_{t-1} : U_{t-1} \to \mathrm{I\!R}$ such that:

$$
p_{t-1} \succeq_{t-1} p'_{t-1} \;\Leftrightarrow\; \mathcal{M}^{f_{t-1}}(p_{t-1}, \tilde{u}_{t-1}) \geq \mathcal{M}^{f_{t-1}}(p'_{t-1}, \tilde{u}_{t-1}) \quad \forall\, p_{t-1}, p'_{t-1} \in P_{t-1}.
$$

Hence H1 also holds for $t-1$. Moreover as shown in the proof of proposition 7 for every lottery $p_{t-1} \in P_{t-1}$ there exists a certainty equivalent $\tilde{x}^{p_{t-1}} = (x_{t-1}^{p_{t-1}}, p_t^{p_{t-1}}) \in X_{t-1} \times P_t$ such that $p_{t-1} \sim_{t-1} \tilde{x}^{p_{t-1}}$. Given that induction hypothesis H2 holds for t, there exists a certain consumption path $\mathsf{x}^{p_t^{p_{t-1}}}$ with $\mathsf{x}^{p_t^{p_{t-1}}} \sim_t p_t^{p_{t-1}}$. Therefore by time consistency $\mathsf{x}^{p_{t-1}} \equiv (x_{t-1}^{p_{t-1}}, \mathsf{x}^{p_t^{p_{t-1}}})$ is a certain consumption path which satisfies $\mathsf{x}^{p_{t-1}} \sim_{t-1} p_{t-1}$. Hence, the second induction hypothesis H2 is satisfied for $t-1$ as well, and recursion gives that H1 and thus the theorem is satisfied for all $t \in \{1, ..., T\}$.

Part IV (“$\Leftarrow$”):

A1 (weak order): Transitivity and completeness are trivial.
A2 (independence): Let $p_t \sim_t p'_t$. Then for any $p''_t \in P_t, a \in [0,1]$ it follows:

$$\begin{array}{rccc} & p_t & \sim_t & p'_t \\ \Leftrightarrow & f_t^{-1} \int f_t \tilde{u}_t \, dp_t & = & f_t^{-1} \int f_t \tilde{u}_t \, dp'_t \\ \Leftrightarrow & \int f_t \tilde{u}_t \, dp_t & = & \int f_t \tilde{u}_t \, dp'_t \\ \Leftrightarrow & a \int f_t \tilde{u}_t \, dp_t + (1-a) \int f_t \tilde{u}_t \, dp''_t & = & a \int f_t \tilde{u}_t \, dp'_t + (1-a) \int f_t \tilde{u}_t \, dp''_t \\ \Leftrightarrow & f_t^{-1} \int f_t \tilde{u}_t \, d(a\, p_t + (1-a)\, p''_t) & = & f_t^{-1} \int f_t \tilde{u}_t \, d(a\, p'_t + (1-a)\, p''_t) \\ \Leftrightarrow & a\, p_t + (1-a)\, p''_t & \sim_t & a\, p'_t + (1-a)\, p''_t . \end{array}$$

A3 (continuity): Using the topology of weak convergence on P_t, the functional $\mathcal{M}^{f_t}(\cdot, \tilde{u}_t) : P_t \to \mathrm{I\!R}$ is continuous. For all $p_t \in P_t$ define the numbers $U^{p_t} \in \mathrm{I\!R}$ by $U^{p_t} = \mathcal{M}^{f_t}(p_t, \tilde{u}_t)$. Then, the sets $\{p'_t \in P_t : p'_t \succeq_t p_t\}$ and $\{p'_t \in P_t : p_t \succeq_t p'_t\}$ are the inverse image of the closed intervals $[U^{p_t}, \overline{U}]$ and $[\underline{U}, U^{p_t}]$ under $\mathcal{M}^{f_t}(\cdot, \tilde{u}_t)$ and as such they are closed.
A4 (certainty additivity): Defining $u^o_\tau = g_\tau \circ u_\tau$ for all $\tau \in \{1, ..., T\}$ find that for all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^T$:

$$\begin{array}{rccc} & \mathsf{x} & \succeq & \mathsf{x}' \\ \Leftrightarrow & \tilde{u}_t(\mathsf{x}) & \geq & \tilde{u}_t(\mathsf{x}') \\ \Leftrightarrow & \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}_\tau) & \geq & \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}'_\tau) \\ \Leftrightarrow & \sum_{\tau=t}^T u^o_\tau(\mathsf{x}_\tau) & \geq & \sum_{\tau=t}^T u^o_\tau(\mathsf{x}'_\tau). \end{array}$$

A5 (time consistency): For all $t \in \{1, ..., T\}$ find for all $x_t \in X_t$ and $p_{t+1}, p'_{t+1} \in P_{t+1}$:

$$\begin{array}{rl} & (x_t, p_{t+1}) \quad \succeq_t \quad (x_t, p'_{t+1}) \\ \Leftrightarrow & g_t^{-1} \left[\theta_t \, g_t \circ u_t(x_t) + \theta_t \theta_{t+1}^{-1} \, g_{t+1} \circ M^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) + \theta_t \theta_{t+1}^{-1} \vartheta_t \right] \\ & \geq g_t^{-1} \left[\theta_t \, g_t \circ u_t(x_t) + \theta_t \theta_{t+1}^{-1} \, g_{t+1} \circ M^{f_{t+1}}(p'_{t+1}, \tilde{u}_{t+1}) + \theta_t \theta_{t+1}^{-1} \vartheta_t \right] \\ \Leftrightarrow & M^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \quad \geq \quad M^{f_{t+1}}(p'_{t+1}, \tilde{u}_{t+1}) \\ \Leftrightarrow & p_{t+1} \quad \succeq_{t+1} \quad p'_{t+1}. \end{array}$$

Moreover Part: Uniqueness of $(f_t)_{t \in \{1 \ldots T\}}$ up to affine transformations follows for each f_t as in the proof of theorem 2.[1] In the following I give the proof for the uniqueness result for $(g_t)_{t \in \{1 \ldots T\}}$ which is slightly more involved.
"$\Leftarrow$": It has to be proven that if the triple $(u_\tau, f_\tau, g_\tau)_{\tau \in \{t, \ldots, T\}}$ represents $\succeq_t$ as in the theorem, then so does the tuple $(u_\tau, f_\tau, g'_\tau)_{\tau \in \{t, \ldots, T\}}$ with $(g'_\tau)_{\tau \in \{t, \ldots, T\}} = (a g_\tau + b_\tau)_{\tau \in \{t, \ldots, T\}}$ for any $a, b \in \mathrm{I\!R}, a > 0$. For the g'_τ-scenario the normalization constants change as

[1] As seen in theorem 2 the allowed transformations for f and g are independent.

follows.

$$\theta'_t = \frac{\Delta G'_t}{\sum_{\tau=t}^T \Delta G'_\tau} = \frac{a\Delta G_t}{\sum_{\tau=t}^T a\Delta G_\tau} = \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau} = \theta_t \quad \text{and} \tag{C.3}$$

$$\begin{aligned}\vartheta'_t &= \frac{\overline{G'}_{t+1}\underline{G'}_t - \underline{G'}_{t+1}\overline{G'}_t}{\Delta G'_t} = \frac{(a\overline{G}_{t+1}+b_{t+1})(a\underline{G}_t+b_t)-(a\underline{G}_{t+1}+b_{t+1})(a\overline{G}_t+b_t)}{a\Delta G_t} \\ &= a\vartheta_t + \frac{b_{t+t}a(\underline{G}_t-\overline{G}_t)+b_t a(\overline{G}_{t+1}-\underline{G}_{t+1})}{a\Delta G_t} + \frac{b_{t+1}b_t - b_{t+1}b_t}{a\Delta G_t} = a\vartheta_t - b_{t+1} + b_t\frac{\Delta G_{t+1}}{\Delta G_t} \end{aligned} \tag{C.4}$$

for $t \in \{1, ..., T\}$.[2] Hence, noting that $g_t'^{-1}(\cdot) = g_t^{-1}\left[a_t^{-1}\{(\cdot) - b_t\}\right]$, the intertemporal aggregation rule transforms as

$$\begin{aligned}\mathcal{N}^{g'_t,g'_{t+1}}(\cdot,\cdot) &= g_t'^{-1}\left[\theta'_t g'_t(\cdot) + \theta'_t\theta'^{-1}_{t+1}g'_{t+1}(\cdot) + \theta'_t\theta'^{-1}_{t+1}\vartheta'_t\right] \\ &= g_t^{-1}\Big[a^{-1}\Big\{\theta_t(ag_t(\cdot)+b_t) + \theta_t\theta^{-1}_{t+1}(ag_{t+1}(\cdot)+b_{t+1}) \\ &\qquad +\theta_t\theta^{-1}_{t+1}(a\vartheta_t - b_{t+1} + b_t\tfrac{\Delta G_{t+1}}{\Delta G_t}) - b_t\Big\}\Big] \\ &= g_t^{-1}\Big[\theta_t g_t(\cdot) + \theta_t\theta^{-1}_{t+1}g_{t+1}(\cdot) + \theta_t\theta^{-1}_{t+1}\vartheta_t + a^{-1} \\ &\qquad \Big\{\theta_t b_t + \theta_t\theta^{-1}_{t+1}b_{t+1} + \theta_t\theta^{-1}_{t+1}(-b_{t+1} + b_t\tfrac{\Delta G_{t+1}}{\Delta G_t}) - b_t\Big\}\Big] \\ &= g_t^{-1}\Big[\theta_t g_t(\cdot) + \theta_t\theta^{-1}_{t+1}g_{t+1}(\cdot) + \theta_t\theta^{-1}_{t+1}\vartheta_t\Big] = \mathcal{N}^{g_t,g_{t+1}}(\cdot,\cdot)\,.\end{aligned}$$

To arrive at the last line I have used equation (C.2). It results that $(u_\tau, f_\tau, g'_\tau)_{\tau\in\{t,...,T\}}$ is a representation of $\succeq_t$.

"⇒": From the proof of the main part (in particular from equation C.1) it is known that if the sequences $(f_\tau)_{\tau\in\{t,...,T\}}$ and $(g_\tau)_{\tau\in\{t,...,T\}}$ represent $\succeq_t$ as in the theorem, then $\sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}_\tau)$ represents $\succeq_t$ on the set of certain outcome paths. In particular assume that $(u_\tau, f_\tau, g_\tau)_{\tau\in\{1,...,T\}}$ and $(u_\tau, f'_\tau, g'_\tau)_{\tau\in\{1,...,T\}}$ both represent $\succeq_1$. Then $\sum_{\tau=1}^T g_\tau \circ u_\tau(\mathsf{x}_\tau)$ and $\sum_{\tau=1}^T g'_\tau \circ u_\tau(\mathsf{x}_\tau)$ both represent $\succeq_1 |_{\mathsf{X}^1}$. Pick any $r \in \{1, ..., T\}$. Define $\tilde{g}_\tau = \frac{\Delta G_r}{\Delta G'_r}g'_\tau + g_\tau(\underline{u}_\tau) - g'_\tau(\underline{u}_\tau)\frac{\Delta G_r}{\Delta G'_r}$ for all $\tau \in \{1, ..., T\}$ which yields

$$\tilde{g}_\tau(\underline{u}_\tau) = g_\tau(\underline{u}_\tau) \quad \forall \tau \in \{1, ..., T\} \quad \text{and}$$
$$\tilde{g}_r(\overline{u}_r) = g_r(\overline{u}_r).$$

As $(\tilde{g}_\tau)_{\tau\in\{1,...,T\}} = (ag'_\tau + b_\tau)_{\tau\in\{1,...,T\}}$ with $a > 0$ it follows from "⇐" that $\sum_{\tau=t}^T \tilde{g}_\tau \circ u_\tau(\mathsf{x}_\tau)$ represents $\succeq_1 |_{\mathsf{X}^1}$. In the following, I will show that in fact $(\tilde{g}_\tau)_{,\tau\in\{1,...,T\}} = (g_\tau)_{\tau\in\{1,...,T\}}$ completing the proof.

The proof that $(\tilde{g}_\tau)_{\tau\in\{1,...,T\}}$ has to be equal to $(g_\tau)_{\tau\in\{1,...,T\}}$ distinguishes two cases. In the first case, I assume that for all $\tau \in \{1, ..., T\}$ it is $\tilde{g}_\tau \neq g_\tau$. In the second case, I assume that for at least one $\tau \in \{1, ..., T\}$ it is $\tilde{g}_\tau = g_\tau$ and for some other it is $\tilde{g}_{\tau'} \neq g_{\tau'}$. Both assumptions will lead to a contradiction. For the rest of this proof I will slightly modify *notation* letting $U_\tau \in \mathbb{R}$ denote utility levels in period τ (and not the range of

[2]Where b_{T+1} and ΔG_{T+1} are treated as zero to render $\vartheta'_T = 0$.

u_τ).

Case 1: Assume that for all $\tau \in \{1, ..., T\}$ it is $\tilde{g}_\tau \neq g_\tau$.

First, observe that due to continuity, connectedness, $\tilde{g}(\underline{U}_r) = g(\underline{U}_r)$ and $\tilde{g}(\overline{U}_r) = g(\overline{U}_r)$, there have to exist U_r^1 and U_r^2 such that $\tilde{g}(U_r^1) = g(U_r^1), \tilde{g}(U_r^2) = g(U_r^2)$ and $\tilde{g}(U_r) \neq g(U_r)\ \forall\, U_r \in (U_r^1, U_r^2)$. Furthermore pick any $s \in \{1, ..., T\}$ with $s \neq r$. Then, again by continuity, connectedness and $\tilde{g}(\underline{U}_s) = g(\underline{U}_s)$ there exist U_s^1 and $\epsilon > 0$ such that $\tilde{g}(U_s) = g(U_s)\ \forall\, U_s \leq U_s^1$ and $\tilde{g}(U_s) \neq g(U_s)\ \forall\, U_s \in (U_s^1, U_s^1 + \epsilon)$. For all $\tau \notin \{s, r\}$ fix $x_\tau = x_\tau^o$ arbitrarily for the rest of this proof. To simplify notation let $(U_s, U_r) \succeq (U_s', U_r')$ be shorthand for $(x_1^o, ..., x_s, ..., x_r, ..., x_T^o) \succeq_1 (x_1^o, ..., x_s', ..., x_r', ..., x_T^o)$, where $U_s = u_s(x_s)$ and $U_r = u_r(x_r)$.[3]

Second, for $U_s^* \in (U_s^1, U_s^1 + \epsilon)$ define U_r^* by the requirement $(U_s^*, U_r^1) \sim (U_s^1, U_r^*)$.[4] Pick a U_s^* for which $U_r^* \in (U_r^1, U_r^2)$. Restating this indifference in the $\sum_{\tau=1}^T \tilde{g}_\tau \circ u_\tau(\mathsf{x}_\tau)$ and $\sum_{\tau=1}^T g_\tau \circ u_\tau(\mathsf{x}_\tau)$ repesentations for $\succeq_1 |_{\mathsf{X}^1}$ yields respectively

$$\begin{aligned} \tilde{g}_s(U_s^*) + \tilde{g}_r(U_r^1) &= \tilde{g}_s(U_s^1) + \tilde{g}_r(U_r^*) \\ g_s(U_s^*) + g_r(U_r^1) &= g_s(U_s^1) + g_r(U_r^*)\,. \end{aligned}$$

The terms stemming from periods $\tau \notin \{r, s\}$ cancel because they are the same on both sides of the equation. Moreover by construction it is $\tilde{g}_r(U_r^1) = g_r(U_r^1)$ and $\tilde{g}_s(U_s^1) = g_s(U_s^1)$. Therefore taking the difference of the two equation above brings about

$$\tilde{g}_s(U_s^*) - g_s(U_s^*) = \tilde{g}_r(U_r^*) - g_r(U_r^*)\,. \tag{C.5}$$

Third define U_r^{**} by $(U_s^*, U_r^{**}) \sim (U_s^1, U_r^2)$. Such U_r^{**} exists and lies in (U_r^1, U_r^2) due to continuity, connectedness and $(U_s^*, U_r^1) \sim (U_s^1, U_r^*) \prec (U_s^1, U_r^2) \sim (U_s^*, U_r^* *)$ and $(U_s^*, U_r^2) \succ (U_s^1, U_r^2) \sim (U_s *, U_r^* *)$. Stating the indifference condition in the $\sum_{\tau=1}^T \tilde{g}_\tau \circ u_\tau(\mathsf{x}_\tau)$ and $\sum_{\tau=1}^T g_\tau \circ u_\tau(\mathsf{x}_\tau)$ representations for $\succeq_1 |_{\mathsf{X}^1}$ yields respectively

$$\begin{aligned} \tilde{g}_s(U_s^*) + \tilde{g}_r(U_r^{**}) &= \tilde{g}_s(U_s^1) + \tilde{g}_r(U_r^2) \\ g_s(U_s^*) + g_r(U_r^{**}) &= g_s(U_s^1) + g_r(U_r^2)\,. \end{aligned}$$

Due to $\tilde{g}_s(U_s^1) = g_s(U_s^1)$ and $\tilde{g}_r(U_r^2) = g_r(U_r^2)$ taking the difference between these two equations renders

$$\tilde{g}_s(U_s^*) - g_s(U_s^*) = -\left[\tilde{g}_r(U_r^{**}) - g_r(U_r^{**})\right]\,. \tag{C.6}$$

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[3]For $r < s$ nothing but the order in this notation would change.

[4]It can happen that for some U_s^* there is no U_r^* high enough within the domain of u_r such that the indifference holds (i.e. no $x_r \in X_r$ is good enough to make up for getting outcome x_s^1 instead of x_s^* in period s). However continuity, connectedness and $U_r^1 < \overline{U}_r$ make sure that for small enough $U_s^* > U_s^1$ there exist U_r^* satisfying the condition. In particular for any U_s^* satisfying the condition given in the next line there obviously exists U_r^*.

The fourth step derives a contradiction from combining equations (C.5) and (C.6). Together these two equations yield the following statement:

$$\tilde{g}_r(U_r^*) - g_r(U_r^*) = -\left[\tilde{g}_r(U_r^{**}) - g_r(U_r^{**})\right] . \tag{C.7}$$

Recollect that $U_r^*, U_r^{**} \in (U_r^1, U_r^2)$ where by construction it has to hold that $\tilde{g}_r(U_r) \neq g_r(U_r)$. Together with equation (C.7) it follows that either $\tilde{g}_r(U_r^*) > g_r(U_r^*)$ and $\tilde{g}_r(U_r^{**}) < g_r(U_r^{**})$ or vice versa. In any case continuity together with connectedness implies that there exists U_r^0 between U_r^* and U_r^{**} for which $\tilde{g}_r(U_r^0) = g_r(U_r^0)$. But this contradicts $\tilde{g}(U_r) \neq g(U_r)\ \forall\, U_r \in (U_r^1, U_r^2)$ and hence the "case 1 assumption" that for all $\tau \in \{1, ..., T\}$ it is $\tilde{g}_\tau \neq g_\tau$.

Case 2: Assume that there exist $i, j \in \{1, ..., T\}$ such that $\tilde{g}_i \neq g_i$ and $\tilde{g}_j = g_j$. Then, in analogy to case 1 there exist U_i^1 and $\epsilon > 0$ such that $\tilde{g}(U_i) = g(U_i)\ \forall\, U_i \leq U_i^1$ and $\tilde{g}(U_i) \neq g(U_i)\ \forall\, U_i \in (U_i^1, U_i^1 + \epsilon)$. Furthermore fix some $U_j^1 < \overline{U}_j$. As in case 1 fix $\tau \notin \{s, r\}$ to some arbitrary x_τ^o and use the shorthand notation $(U_i, U_j) \succeq (U_i', U_j')$ for $(x_1^o, ..., x_i, ..., x_j, ..., x_T^o) \succeq_1 (x_1^o, ..., x_i', ..., x_j', ..., x_T^o)$.

Pick some $U_i^* \in (U_i^1, U_i^1 + \epsilon)$ such that there exists $U_j^* < \overline{U}_j$ satisfying $(U_i^*, U_j^1) \sim (U_i^1, U_j^*)$.[5] Stating this indifference condition in the $\sum_{\tau=1}^T \tilde{g}_\tau \circ u_\tau(\mathsf{x}_\tau)$ and $\sum_{\tau=1}^T g_\tau \circ u_\tau(\mathsf{x}_\tau)$ repesentations for $\succeq_1 |_{\mathsf{X}^1}$ yields respectively

$$\begin{aligned}
\tilde{g}_i(U_i^*) + \tilde{g}_j(U_j^1) &= \tilde{g}_i(U_i^1) + \tilde{g}_j(U_j^*)\\
g_i(U_i^*) + g_j(U_j^1) &= g_i(U_i^1) + g_j(U_j^*) .
\end{aligned}$$

Again the terms stemming from periods $\tau \notin \{i, j\}$ cancel. Moreover it is $\tilde{g}_j(U_j^1) = g_j(U_j^1)$, $\tilde{g}_i(U_i^1) = g_i(U_i^1)$ and $\tilde{g}_j(U_j^*) = g_j(U_j^*)$. But this implies $\tilde{g}_i(U_i^*) = g_i(U_i^*)$ in contradiction to $\tilde{g}(U_i) \neq g(U_i)\ \forall\, U_i \in (U_i^1, U_i^1 + \epsilon)$. □

Proof of theorem 5: The moreover part of theorem 4 shows that the functions g_t for $t \in \{1, ..., T\}$ exhibit an affine freedom with translational constants b_t^g that are independent between the different periods. In the following I show that this freedom can be used to eliminate the normalization constants ϑ_t in the representation of theorem 4.
"$\Rightarrow$": Take an arbitrary representation $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ for the preferences $(\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 4. I construct a transformation $g_t \to \tilde{g}_t = g_t + b_t$, such that the particular representation going along with $(u_t, f_t, \tilde{g}_t)_{t\in\{1,...,T\}}$ satisfies $\tilde{\vartheta}_t = 0$ for all $t \in \{1, ..., T\}$. Let the translation parameter b_1, $g_1 \to \tilde{g}_1 = g_1 + b_1$, be arbitrary but

[5] Again the existence of an appropriate U_j^1 is just a question of picking $U_i^* \in (U_i^1, U_i^1 + \epsilon)$ close enough to U_i^0.

fixed. Equation (C.4) shows that the condition for $\tilde{\vartheta}_t = 0$ is equivalent to

$$\begin{aligned} \tilde{\vartheta}_t &= a\vartheta_t - b_{t+1} + b_t \tfrac{\Delta G_{t+1}}{\Delta G_t} \stackrel{!}{=} 0 \\ \Leftrightarrow b_{t+1} &= a\vartheta_t + b_t \tfrac{\Delta G_{t+1}}{\Delta G_t} \,. \end{aligned} \tag{C.8}$$

Fixing inductively the constants b_t for $t > 1$ by equation (C.8) eliminates the normalization constants $\tilde{\vartheta}_t$ and renders the new intertemporal aggregation rule

$$\begin{aligned} \mathcal{N}_*^{g_t,g_{t+1}} &: \; U_t \times U_{t+1} \to \mathrm{I\!R} \\ \mathcal{N}_*^{g_t,g_{t+1}}(\cdot,\cdot) &= \tilde{g}_t^{-1}\left[\theta_t \tilde{g}_t(\cdot) + \theta_t \theta_{t+1}^{-1} \tilde{g}_{t+1}(\cdot)\right] \,, \end{aligned}$$

which is used in the representation of theorem 5.
"$\Leftarrow$": Observe that the condition $\frac{\overline{G}_{t+1}}{\underline{G}_{t+1}} = \frac{\overline{G}_t}{\underline{G}_t}$ implies $\vartheta_t = 0$:

$$\begin{aligned} \frac{\overline{G}_{t+1}}{\underline{G}_{t+1}} &= \frac{\overline{G}_t}{\underline{G}_t} \\ \Leftrightarrow \overline{G}_{t+1}\underline{G}_t &= \overline{G}_t \underline{G}_{t+1} \\ \Leftrightarrow \overline{G}_{t+1}\underline{G}_t - \overline{G}_t \underline{G}_{t+1} &= 0 \\ \Leftrightarrow \quad \vartheta_t &= 0 \,. \end{aligned}$$

But then, the necessity part of the proof is a special case of theorem 4.
Moreover part: "$\Rightarrow$": Let $(u_t, f_t, g_t)_{t\in\{1,\dots,T\}}$ and $(u_t, f'_t, g'_t)_{t\in\{1,\dots,T\}}$ be representations in the sense of theorem 5. By theorem 4, I know that the functions g_t and g'_t can at most differ by an affine translation of type $g_t \to g'_t = a\, g_t + b_t$. In addition equation (C.8) has to be satisfied with $\vartheta_t = 0$ for all $t \in \{1, \dots, T-1\}$. Given b_1 and defining $b = \frac{b_1}{\Delta G_1}$, the latter equation determines all translation parameters b_t as

$$\begin{aligned} b_t &= b_{t-1}\tfrac{\Delta G_t}{\Delta G_{t-1}} = b_{t-2}\tfrac{\Delta G_{t-1}}{\Delta G_{t-2}}\,\tfrac{\Delta G_t}{\Delta G_{t-1}} = b_{t-2}\tfrac{\Delta G_t}{\Delta G_{t-2}} = b_{t-3}\tfrac{\Delta G_t}{\Delta G_{t-3}} = \dots \\ &= b_1 \tfrac{\Delta G_t}{\Delta G_1} = b\,\Delta G_1 \,\tfrac{\Delta G_t}{\Delta G_1} = \Delta G_t\, b \,. \end{aligned}$$

Therefore, the remaining freedom of a representation $(u_t, f_t, g_t)_{t\in\{1,\dots,T\}}$ in the sense of theorem 5 is given by the simultaneous transformations $g_t \to g'_t = a\, g_t + \Delta G_t\, b$ for all $t \in \{1, \dots, T\}$ with $a \in \mathrm{I\!R}_{++}$ and $b \in \mathrm{I\!R}$ and, as before, the independent transformations $f_t \to f'_t = a^f_t\, f_t + b^f_t$ for $t \in \{1, \dots, T\}$ with $a^f_t \in \mathrm{I\!R}_{++}$ and $b^f_t \in \mathrm{I\!R}$.
"$\Leftarrow$": Wlog let $(u_t, f_t, g_t)_{t\in\{1,\dots,T\}}$ be a representation of $(\succeq_t)_{t\in\{1,\dots,T\}}$ in the sense of theorem 5. Recalling from the main part of the proof that $\frac{\overline{G}_{t+1}}{\underline{G}_{t+1}} = \frac{\overline{G}_t}{\underline{G}_t}$ implies $\vartheta_t = 0$, it is know that $(u_t, f_t, g_t)_{t\in\{1,\dots,T\}}$ is also a representation in the sense of theorem 4. Thus, by the moreover part theorem 4, $(u_t, f'_t, ag_t + \Delta G_t b)_{t\in\{1,\dots,T\}}$ is a representation in the sense of theorem 4 as well, whenever f'_t is a positive affine transformation of f_t. Moreover, because of $\vartheta_t = 0$ equation (C.4) implies

$$\vartheta'_t = a\vartheta_t - b_{t+1} + b_t \tfrac{\Delta G_{t+1}}{\Delta G_t} = -\Delta G_{t+1}\, b + \Delta G_t\, b\, \tfrac{\Delta G_{t+1}}{\Delta G_t} = 0 \,.$$

Therefore, the sequence of triples $(u_t, f'_t, ag_t + \Delta G_t b)_{t\in\{1,\dots,T\}}$ also is a representation in the sense of theorem 5.

Remark: On certain consumption paths the representation simplifies to the form

$$\tilde{u}_t(\mathsf{x}^t) = g_t^{-1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau \circ u_\tau(\mathsf{x}_\tau^t)\right] . \tag{C.9}$$

This is an immediate consequence of equation (C.1) in the proof of theorem 4. Recognizing that $\vartheta_t = 0$ for all $t \in \{1, ..., T\}$ implies $\xi_t = 0$ in all periods, the referenced equation becomes $\theta_t^{-1}\, g_t \circ \tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^{T} g_\tau \circ u_\tau(\mathsf{x}_\tau^t)$ which is equivalent to equation (C.9).

□

Proof of lemma 4: Due to backwards recursion in the construction of the aggregate utility functions $\tilde{u}_t$, the representing functions for $t > \tau$ stay unaltered by the change $(u_\tau, f_\tau, g_\tau) \rightarrow (u'_\tau, f'_\tau, g'_\tau)$. For aggregate intertemporal utility in period τ find:

$$\begin{aligned}\tilde{u}'_\tau(x_\tau, p_{\tau+1}) &= s_\tau g_\tau^{-1}\left[\theta_\tau\, g_\tau s_\tau^{-1} s_\tau u_\tau(x_\tau) + \theta_\tau \theta_{\tau+1}^{-1}\, g_{\tau+1}\mathcal{M}^{f_{\tau+1}}(p_{\tau+1}, \tilde{u}_{\tau+1}) + \theta_\tau \theta_{\tau+1}^{-1}\vartheta_\tau\right] \\ &= s_\tau \circ \tilde{u}_\tau(x_\tau, p_{\tau+1}) .\end{aligned}$$

Note that the normalization constants do not change as the range of g_τ and g'_τ are the same. Then, the new representing functional in period τ becomes

$$\begin{aligned}\mathcal{M}^{f'_\tau}(p_\tau, \tilde{u}'_\tau) &= \mathcal{M}^{f'_\tau}(p_\tau, s_\tau \circ \tilde{u}_\tau) \\ &= s_\tau \circ f_\tau^{-1}\left[\int f_\tau \circ s_\tau^{-1} \circ s_\tau \circ \tilde{u}_\tau \, dp_\tau\right] = s_\tau \circ \mathcal{M}^{f_\tau}(p_\tau, \tilde{u}_\tau).\end{aligned}$$

As it corresponds to a strictly increasing transformation of $\mathcal{M}^{f_\tau}(p_\tau, \tilde{u}_\tau)$, it represents $\succeq_\tau$. For period $\tau - 1$ the new aggregate intertemporal utility function becomes:

$$\begin{aligned}\tilde{u}'_{\tau-1}(x_{\tau-1}, p_\tau) &= g_{\tau-1}^{-1}\left[\theta_{\tau-1}\, g_{\tau-1} u_{\tau-1}(x_{\tau-1}) + \tfrac{\theta_{\tau-1}}{\theta_\tau}\, g_\tau s_\tau^{-1}\mathcal{M}^{f'_\tau}(p_\tau, \tilde{u}'_\tau) + \tfrac{\theta_{\tau-1}}{\theta_\tau}\vartheta_{\tau-1}\right] \\ &= g_{\tau-1}^{-1}\left[\theta_{\tau-1}\, g_{\tau-1} u_{\tau-1}(x_{\tau-1}) + \tfrac{\theta_{\tau-1}}{\theta_\tau}\, g_\tau s_\tau^{-1} s_\tau \mathcal{M}^{f_\tau}(p_\tau, \tilde{u}_\tau) + \tfrac{\theta_{\tau-1}}{\theta_\tau}\vartheta_{\tau-1}\right] \\ &= \tilde{u}_{\tau-1}(x_{\tau-1}, p_\tau).\end{aligned}$$

Therefore, again due to backwards recursion in the construction of the functions $\tilde{u}_t$, the representing functions in periods $t < \tau$ stay unchanged. In consequence, $(u_t, f_t, g_t)'_{t\in\{1,...,T\}}$ is a representation of $(\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 4, whenever $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ is a representation of $(\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 4. □

Proof of corollary 6: The main part of the proof imitates that of corollary 2. Instead of applying lemma 1, gauge lemma 4 is used for every period.

Moreover part: "⇒": Also the moreover part works along the lines of corollary 2. Let $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ and $(u'_t, f_t, g'_t)_{t\in\{1,...,T\}}$ be representations in the sense of the corollary. Then it exists for every t a strictly increasing, continuous function s_t such that

$u'_t = s_t \circ u_t$. Lemma 4 implies that with $(u'_t, f_t, g'_t)_{t\in\{1,...,T\}} = (s_t\, u_t, f_t, g'_t)_{t\in\{1,...,T\}}$ being a representation of $\succeq$, so is the sequence of triples $(u, f_t s_t, g'_t s_t)_{t\in\{1,...,T\}}$. Comparing the latter to the representation $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ it can be deduced from the moreover part of theorem 4 that there have to exist $a \in \mathbb{R}_{++}$ as well as affine transformations $\mathfrak{a}^+_t \in \mathsf{A}^+$ and $\mathfrak{a}^a_t \in \mathsf{A}^a$ for all $t \in \{1, ..., T\}$, such that

$$f_t = \mathfrak{a}^+_t f_t\, s_t \quad \Leftrightarrow \quad s_t^{-1} = f_t^{-1} a^+_t f_t \quad \text{and} \tag{C.10}$$

$$g_t = \mathfrak{a}^a_t g'_t s_t. \tag{C.11}$$

Substituting the result for the functions s_t into the equations for g_t and u_t renders

$$g_t = \mathfrak{a}^a_t g'_t f_t^{-1} a_t^{+^{-1}} f_t \quad \text{and}$$
$$u_t = s_t^{-1} u'_t = f_t^{-1} a^+_t f_t\, u'_t.$$

"$\Leftarrow$": Let the sequence of triples $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ be a representation of the preferences described by $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$. Then by theorem 4, with $\mathfrak{a}^+_t \in \mathsf{A}^+$ for $t \in \{1, ..., T\}$, and $\mathfrak{a}^a_t \in \mathsf{A}^a$, also the sequence $(u_t, \mathfrak{a}^+_t f_t, \mathfrak{a}^a_t\, g_t)_{t\in\{1,...,T\}}$ is a representation of $\succeq$. By lemma 4 it follows that also $([\mathfrak{a}^+_t f] u_t, \mathfrak{a}^+_t f_t [\mathfrak{a}^+_t f_t]^{-1}, \mathfrak{a}^a_t g_t [\mathfrak{a}^+_t f_t]^{-1})_{t\in\{1,...,T\}} = (\mathfrak{a}^+_t f_t u_t, \mathrm{id}, \mathfrak{a}^a_t g_t f_t^{-1} \mathfrak{a}_t^{+^{-1}})_{t\in\{1,...,T\}}$ is a representation of $\succeq$. Applying lemma 4 once again yields the result that the sequence $(f_t^{-1} \mathfrak{a}^+_t f_t u_t, f_t, \mathfrak{a}^a_t g_t f_t^{-1} \mathfrak{a}_t^{+^{-1}} f_t)_{t\in\{1,...,T\}}$ is a representation of $\succeq$. □

Proof of corollary 7: Imitates the proof of corollary 6. In the moreover part instead of equations (C.10) and (C.11) find

$$f_t = \mathfrak{a}^+_t f'_t\, s_t \quad \text{and}$$
$$g_t = \mathfrak{a}^a_t g_t s_t \quad \Leftrightarrow \quad s_t^{-1} = g_t^{-1} \mathfrak{a}^a_t g_t\,.$$

Substituting the result for the functions s_t into the equations for f_t and u_t renders

$$f_t = \mathfrak{a}^+_t f'_t\, g_t^{-1} \mathfrak{a}_t^{a^{-1}} g_t \quad \text{and}$$
$$u_t = s_t^{-1} u'_t = g_t^{-1} \mathfrak{a}^a_t\, g_t\, u'_t\,.$$

□

Proof of theorem 6: The proof is divided into five parts. In the first, I translate axiom A6$^{\mathrm{s}}$ into the representation of theorem 4. In the second part, I show that the equation derived in the first part locally implies strict concavity of $f_t \circ g_t^{-1}$. Part three extends this result to strict concavity on the entire set Γ_t. Part four proofs the necessity of axiom A6$^{\mathrm{s}}$ for the strict concavity of $f_t \circ g_t^{-1}$. Together, parts one through four proof assertion a) of the theorem for the case of strict intertemporal risk aversion. For the case of strict intertemporal risk seeking just change the signs in the inequalities and replace concave by convex. Part five lays out how assertions b-d) follow from the proof of assertion a). **Part I ("⇒"):** In part one I translate axiom A6^{s} into the representation of theorem 4. I start with the first line, i.e the premise, and use equation (C.1) to find:

$$\begin{array}{ccc} \mathrm{x}^t & \sim_t & \mathrm{x}'^t \\ \Rightarrow g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t\Big] & = & g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}'^t_\tau) + \xi_t\Big]. \end{array} \tag{C.12}$$

The existence of $\tau \in \{t, ..., T\}$ such that $[\mathrm{x}^t_\tau]_\tau \not\sim_\tau [\mathrm{x}'^t_\tau]_\tau$, translates into

$$g_\tau u_\tau(\mathrm{x}^t_\tau) \neq g_\tau u(\mathrm{x}'^t_\tau) \text{ for some } \tau \in \{t, ..., T\}. \tag{C.13}$$

The second line of axiom A6^{s} becomes

$$\begin{array}{lcl} \mathrm{x}^t & \succ_T & \sum_{i=t}^{T} \frac{1}{T-t+1}\,(\mathrm{x}^t_{-i}\mathrm{x}'^t_i). \end{array}$$

$$\Rightarrow g_t^{-1}\Big[\theta_t \textstyle\sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t\Big] >$$

$$f_t^{-1}\Big[\textstyle\sum_{i=t}^{T} \frac{1}{T-t+1} f_t g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\big) + \xi_t\Big]\Big]$$

$$\Rightarrow f_t g_t^{-1}\Big[\theta_t \textstyle\sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t\Big] > \sum_{i=t}^{T} \frac{1}{T-t+1} f_t g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\big) + \xi_t\Big].$$

Using equation (C.12) the left hand side can be transformed as follows:

$$f_t g_t^{-1}\left[\frac{T-t}{T-t+1}\Big[\theta_t \textstyle\sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t\Big] + \frac{1}{T-t+1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}'^t_\tau) + \xi_t\Big]\right] >$$

$$\textstyle\sum_{i=t}^{T} \frac{1}{T-t+1} f_t g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\big) + \xi_t\Big]$$

$$\Rightarrow f_t g_t^{-1}\left[\frac{1}{T-t+1}\Big[\theta_t \textstyle\sum_{i=t}^{T}\sum_{\tau=t}^{T} g_\tau u_\tau\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\big) + \xi_t\Big]\right] >$$

$$\textstyle\sum_{i=t}^{T} \frac{1}{T-t+1} f_t g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\big) + \xi_t\Big]$$

$$\Rightarrow f_t g_t^{-1}\left[\textstyle\sum_{i=t}^{T} \frac{1}{T-t+1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\big) + \xi_t\Big]\right] > \tag{C.14}$$

$$\textstyle\sum_{i=t}^{T} \frac{1}{T-t+1} f_t g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\big) + \xi_t\Big].$$

Let me define the function $\tilde{z} : \mathsf{X}^t \to \Gamma_t$ by $\tilde{z}(\mathrm{x}^t) = \theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t$. Compare part two of the proof of theorem 4 to see that, when restricting the domain to those consumption paths satisfying equation (C.13),[6] the function $\tilde{z}$ is onto $\Gamma_t = \big(\underline{G}_\tau\,,\,\overline{G}_\tau\big) = \Big(\theta_t \sum_{\tau=t}^{T} \underline{G}_\tau + \xi_t\,,\,\theta_t \sum_{\tau=t}^{T} \overline{G}_\tau + \xi_t\Big)$. In particular define $z_i = \tilde{z}\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)\big)$. In this notation equation (C.14) becomes

$$f_t g_t^{-1}\Big(\textstyle\sum_{i=t}^{T} \frac{1}{T-t+1}\, z_i\Big) \quad > \quad \sum_{i=t}^{T} \frac{1}{T-t+1} f_t g_t^{-1}(z_i). \tag{C.15}$$

If equation (C.15) had to hold for all $z_i \in \Gamma_t$ it would be a straight forward condition for strict concavity of $f_t \circ g_t^{-1}$. However, axiom A6^{s} does not immediately imply that the equation has to be met for every choice $(z_i)_{i\in\{t,...,T\}}$, $z_i \in \Gamma_t$. *Equation* (C.15) has to

[6] It is for the latter restriction that the theorem is considering the *open* set Γ_t.

hold *only for sequences* $(z_i)_{i\in\{t,...,T\}}$ that are stemming from consumption paths $(\mathsf{x}^t_{-i}\mathsf{x}'^t_i)$ for which $\mathsf{x}'^t \in \mathsf{X}^t$ and $\mathsf{x}^t_\tau \in \mathsf{X}^t$ *satisfy the premise of axiom A6*$^{\mathrm{s}}$. *In what follows, I proceed to show that this restricted demand is enough to imply strict concavity of* $f_t \circ g_t^{-1}$ *on* Γ_t.

Part II ("⇒"): Let $z^o \in \Gamma_t$. In this part I show that for every such z^o there exists an open neighborhood $N_{z^o} \subset \Gamma_t$ such that equation (C.15) implies strict concavity of $f_t \circ g_t^{-1}$ on N_{z^o}.

In the first step I define a certain consumption path $\mathsf{x}^{ot} \in \mathsf{X}^t$ with $\tilde{z}(\mathsf{x}^{ot}) = z^o$. It will satisfy the additional characteristic that none of its outcomes is extremal. Define $(G^o_\tau)_{\tau\in\{t,...,T\}}$ to be a sequence with $\underline{G}_\tau < G^o_\tau < \overline{G}_\tau \forall\tau$ and $\theta_t \sum_{\tau=t}^T G^o_\tau + \xi_t = z^o$. Such a sequence has to exist as $z^o \in \Gamma_t$ implies $\theta_t \sum_{\tau=t}^T \underline{G}_\tau + \xi_t < z^o < \theta_t \sum_{\tau=t}^T \overline{G}_\tau + \xi_t$. Moreover by connectedness of X and continuity of $g_\tau \circ u_\tau$ there exists for every $\tau \in \{t, ..., T\}$ an outcome $x^o_\tau \in u_\tau^{-1}\left[g_\tau^{-1}\left(G^o_\tau\right)\right]$ such that $G^o_\tau = u_\tau g_\tau(x^o_\tau)$. Define $\mathsf{x}^{ot}_\tau = (x^o_t, ..., x^o_T)$.

In the second step I define deviation paths $\mathsf{x}^{\mu t}$ around x^{ot}. Set $\epsilon_\tau = \min\{G^o_\tau - \underline{G}_\tau, \overline{G}_\tau - G^o_\tau\}$ for $\tau \in \{t, ..., T\}$ and let $\epsilon = \min_{\tau\in\{t,...,T\}} \epsilon_\tau$. By construction of x^{ot} it is $\epsilon > 0$. For *any* sequence $\mu = (\mu_\tau)_{\tau\in\{t,...,T\}}$ with $\mu_\tau \in (-\epsilon, \epsilon)$ define $G^\mu_\tau = G^o_\tau + \mu_\tau$ for all $\tau \in \{t, ..., T\}$. Then each G^μ_τ is element of $(G^o_\tau - \epsilon, G^o_\tau + \epsilon) \subset (\underline{G}_\tau, \overline{G}_\tau)$ and hence there exists $x^{\mu t}_\tau \in u_t^{-1}\left[g_t^{-1}(G^\mu_\tau)\right]$. Define $\mathsf{x}^{\mu t} = (x^\mu_t, ..., \mathsf{x}^\mu_T)$.

Third, I calculate the $z^\mu_i \in \Gamma_t$ corresponding to the consumption paths $(\mathsf{x}^{ot}_{-i}\mathsf{x}^{\mu t}_i)$ and restate the condition $x^{ot} \sim_t x^{\mu t}$ in terms of z^o and $(z^\mu_i)_{i\in\{t,...,T\}}$. It is

$$\begin{aligned} z^\mu_i = \tilde{z}\big((\mathsf{x}^{ot}_{-i}\mathsf{x}^{\mu t}_i)\big) &= \theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau \left((\mathsf{x}^{ot}_{-i}\mathsf{x}^{\mu t}_i)_\tau\right) + \xi_t \\ &= \theta_t \left((\textstyle\sum_{\tau=t}^T G^o_\tau) - G^o_i + G^\mu_i\right) + \xi_t \\ &= z^o + \theta_t(G^\mu_i - G^o_i). \end{aligned}$$

Hence $z^\mu_i = \tilde{z}\big((\mathsf{x}^{ot}_{-i}\mathsf{x}^{\mu t}_i)\big)$ as a fuction of μ_i is *onto* $(G^o_\tau - \theta_t\epsilon\,,\, G^o_\tau + \theta_t\epsilon)$. The equation also implies that the condition $[\mathsf{x}^{ot}_\tau] \not\sim_\tau [\mathsf{x}^{\mu t}_\tau] \Leftrightarrow g_\tau u_\tau(x^\eta_\tau) \neq g_\tau u(x^\mu_\tau) \Leftrightarrow G^o_\tau \neq G^\mu_\tau$ for some $\tau \in \{t, ..., T\}$ is equivalent $z^\mu_i \neq z^o$ for some τ. Using equation (C.12) I further find that $x^{ot} \sim_t x^{\mu t}$ translates into

$$\begin{aligned} &\theta_t \textstyle\sum_{\tau=t}^T G^o_\tau + \xi_t = \theta_t \sum_{\tau=t}^T G^\mu_\tau + \xi_t \\ \Rightarrow\ &\theta_t \textstyle\sum_{\tau=t}^T G^o_\tau + \xi_t = \frac{T-t}{T-t+1}\left(\theta_t \sum_{\tau=t}^T G^o_\tau + \xi_t\right) + \frac{1}{T-t+1}\left(\theta_t \sum_{\tau=t}^T G^\mu_\tau + \xi_t\right) \\ \Rightarrow\ &\theta_t \textstyle\sum_{\tau=t}^T G^o_\tau + \xi_t = \frac{1}{T-t+1}\sum_{i=t}^T \left(\theta_t \left((\sum_{\tau=t}^T G^o_\tau) - G^o_i + G^\mu_i\right) + \xi_t\right) \\ \Rightarrow\ &\qquad\quad z^o \qquad\quad = \frac{1}{T-t+1}\textstyle\sum_{i=t}^T z^\mu_i. \end{aligned}$$

Summarizing steps one to three I have shown that equation (C.15) has to hold for all sequences $(z_i)_{i\in\{t,...,T\}}$ with $z_i \in (z^o - \theta_t\epsilon, z^o + \theta_t\epsilon)$ satisfying $\frac{1}{T-t+1}\sum_{i=t}^T z_i = z^o$ (and not all $z_i = z^o$). However, due to the restriction that the weighted average has to equal z^o

this requirement is not enough to guarantee concavity of $f_t g_t^{-1}$ on $z_i \in (z^o - \theta_t \epsilon, z^o + \theta_t \epsilon)$. *Define* $N_{z^o} = (z^o - \frac{\theta_t \epsilon}{2}, z^o + \frac{\theta_t \epsilon}{2})$. In the following I proceed to show that (C.15) has to hold for *all* non-constant sequences $(z_i)_{i \in \{t,...,T\}}$ with $z_i \in N_{z^o}$. The latter will be sufficient to guarantee strict concavity of $f_t g_t^{-1}$ on the open set N_{z^o}.

In step four, take any $z^* \in N_{z^o}$. I construct a corresponding consumption path x^{*t} with $z^* = \tilde{z}(\mathrm{x}^{*t})$ as well as a perturbation $\mathrm{x}^{\eta t}$ around it. Define

$$G_\tau^* = G_\tau^o + \tfrac{z^* - z^o}{\theta_t (T-t+1)} \in \left(G_\tau^o - \tfrac{-\theta_t \epsilon}{2\theta_t (T-t+1)}, G_\tau^o + \tfrac{-\theta_t \epsilon}{2\theta_t (T-t+1)}\right) \subset (G_\tau^o - \epsilon, G_\tau^o + \epsilon).$$

Then there exists $x_\tau^* \in u_t^{-1}\left[g_t^{-1}(G_\tau^*)\right]$. Define the consumption path $\mathrm{x}^{*t} = (x_t^*, ..., x_T^*)$ and find that indeed

$$\begin{aligned} \tilde{z}(\mathrm{x}^{*t}) &= \theta_t \sum_{\tau=t}^{T} G_\tau^* + \xi_t = \theta_t \left(\sum_{\tau=t}^{T} G_\tau^o + \tfrac{z^* - z^o}{\theta_t (T-t+1)} \right) + \xi_t \\ &= z^o + z^* - z^o \left(\sum_{\tau=t}^{T} \tfrac{1}{(T-t+1)} \right) = z^* \end{aligned}$$

Aim of the following construction is to make sure that the perturbations $\mathrm{x}^{\eta t}$ around x^{*t} account for all sequences $(z_i)_{i \in \{t,...,T\}}$ with $z_i \in N_{z^o}$ that satisfy $\frac{1}{T-t+1} \sum_{i=t}^{T} z_i = z^*$. Define $\epsilon_-^* = \epsilon - (G_i^o - G_i^*)$ and $\epsilon_+^* = \epsilon + (G_i^o - G_i^*)$. For any sequence $\eta = (\eta_\tau)_{\tau \in \{t,...,T\}}$ with $\eta_\tau \in (-\epsilon_-^*, \epsilon_+^*)$ let $G_\tau^\eta = G_\tau^o + \eta_\tau$ for all $\tau \in \{t, ..., T\}$. Then each G_τ^η is in $(G_\tau^o - \epsilon, G_\tau^o + \epsilon) \subset (\underline{G}_\tau, \overline{G}_\tau)$ and hence there exists $x_\tau^{\eta t} \in u_t^{-1}\left[g_t^{-1}(G_\tau^\eta)\right]$. Let $\mathrm{x}^{\eta t} = (x_t^\eta, ..., \mathrm{x}_T^\eta)$.

In step five, I calculate the $z_i^\eta = \tilde{z}\big((\mathrm{x}^{*t}_{-i} \mathrm{x}_i^{\eta t})\big)$ corresponding to the consumption paths $(\mathrm{x}^{*t}_{-i} \mathrm{x}_i^{\eta t})$ and restate the condition $x^{*t} \sim_t x^{\eta t}$ in terms of z^* and $(z_i^\eta)_{i \in \{t,...,T\}}$. It is

$$\begin{aligned} z_i^\eta = \tilde{z}\big((\mathrm{x}^{*t}_{-i} \mathrm{x}_i^{\eta t})\big) &= \theta_t \textstyle\sum_{\tau=t}^{T} g_\tau u_\tau \left((\mathrm{x}^{*t}_{-i} \mathrm{x}_i^{\eta t})_\tau \right) + \xi_t \\ &= \theta_t \left((\textstyle\sum_{\tau=t}^{T} G_\tau^*) - G_i^* + G_i^\eta \right) + \xi_t \\ &= z^* + \theta_t (G_i^\eta - G_i^*). \end{aligned} \tag{C.16}$$

As before with x^{ot} and $x^{\mu t}$ the condition $[\mathrm{x}^{*t}_\tau] \not\sim_\tau [\mathrm{x}^{\eta t}_\tau]$ for some $\tau \in \{t, ..., T\}$ is equivalent to $z_i^\mu \neq z^o$ for some i and equations (C.12) and (C.16) translate $x^{*t} \sim_t x^{\eta t}$ into

$$z^* = \tfrac{1}{T-t+1} \textstyle\sum_{i=t}^{T} z_i^\eta \,.$$

In step six it is shown that the z_i^η calculated in the previous step can generate any sequence $(z_i)_{i \in \{t,...,T\}}$ with elements $z_i \in N_{z^o}$ that satisfies $\frac{1}{T-t+1} \sum_{i=t}^{T} z_i^\eta = z^*$. To verify this fact find that each $z_i^\eta = z^* + \theta_t(G_i^\eta - G_i^*)$ can take any[7] of the values in

$$\begin{aligned}
&\Big(z^* + \theta_t(-\epsilon_-^*)\ ,\ z^* + \theta_t\epsilon_-^*\Big)\\
=\ &\Big(z^o + (z^* - z^o) - \theta_t\big(\epsilon - (G_i^o - G_i^*)\big)\ ,\ z^o + (z^* - z^o) + \theta_t\big(\epsilon + (G_i^o - G_i^*)\big)\Big)\\
=\ &\Big(z^o + (z^* - z^o) - \theta_t\epsilon - \theta_t\tfrac{z^*-z^o}{\theta_t(T-t+1)}\ ,\ z^o + (z^* - z^o) + \theta_t\epsilon - \theta_t\tfrac{z^*-z^o}{\theta_t(T-t+1)}\Big)\\
=\ &\Big(z^o - \theta_t\epsilon + (z^* - z^o)\left(1 - \tfrac{1}{T-t+1}\right)\ ,\ z^o + \theta_t\epsilon + (z^* - z^o)\left(1 - \tfrac{1}{T-t+1}\right)\Big)
\end{aligned}$$

which due to $z^* \in N_{z^o} = (z^o - \frac{\theta_t\epsilon}{2}, z^o + \frac{\theta_t\epsilon}{2})$ is a superset of

$$\begin{aligned}
\supseteq\ &\Big(z^o - \theta_t\epsilon + \tfrac{\theta_t\epsilon}{2}\left(1 - \tfrac{1}{T-t+1}\right)\ ,\ z^o + \theta_t\epsilon - \tfrac{\theta_t\epsilon}{2}\left(1 - \tfrac{1}{T-t+1}\right)\Big)\\
\supseteq\ &\Big(z^o - \tfrac{\theta_t\epsilon}{2}\ ,\ z^o + \tfrac{\theta_t\epsilon}{2}\Big).
\end{aligned}$$

Therefore the z_i^η can take on any value in N_{z^o} as long as the sequence satisfies $z^* = \frac{1}{T-t+1}\sum_{i=t}^T z_i^\eta$. Hence equation (C.15) also has to hold for all non-constant sequences $(z_i)_{i\in\{t,\ldots,T\}}$ with $z_i \in N_{z^o}$ and $\frac{1}{T-t+1}\sum_{i=t}^T z_i = z^*$.

Finally, I show that $f_t g_t^{-1}$ has to be strictly concave on N_{z^o}. Equation (C.15) has to hold for all non-constant sequences $(z_i)_{i\in\{t,\ldots,T\}}$ with $z_i \in N_{z^o}$ and $\frac{1}{T-t+1}\sum_{i=t}^T z_i = z^*$. But z^* was an arbitrary element of N_{z^o} and steps four to six hold for any $z^* \in N_{z^o}$. Therefore equation (C.15) has to hold for all sequences $(z_i)_{i\in\{t,\ldots,T\}}$ with $z_i \in N_{z^o}$ except for the constant sequences with $z_i = z_j\ \forall\, i, j \in \{t, ..., T\}$).[8] Now pick any $l \in \{t, ..., T-1\}$ and define $\lambda = \frac{l-t+1}{T-t+1} > 0$. Furthermore for any pair $z_a, z_b \in N_{z^o}$ select $z_t = ... = z_l = z_a$ and $z_{l+1} = ... = z_T = z_b$. Then equation (C.15) becomes

$$f_t g_t^{-1}\big(\lambda z_a + (1-\lambda)z_b\big) \quad > \quad \lambda f_t g_t^{-1}(z_a) + (1-\lambda) f_t g_t^{-1}(z_b)$$

and has to hold for all $z_a, z_b \in N_{z^o}, z_a \neq z_b$. But due to the continuity of $f_t \circ g_t^{-1}$ this implies strict concavity of $f_t \circ g_t^{-1}$ on N_{z^o} (Hardy et al. 1964, 74,75).

Part III ("⇒"): In this part I show that the local strict concavity of $f_t \circ g_t^{-1}$ on N_{z^o} for all $z^o \in N_{z^o}$ as derived in the second part implies strict concavity on Γ_t.[9] I will first demonstrate that weak concavity extends to Γ_t and then that local strict concavity together with global weak concavity imply strict concavity of $f_t \circ g_t^{-1}$ on all of Γ_t.

First, note that a concave function $h_t = f_t \circ g_t^{-1}$ on N_{z^o} has non-increasing right-continuous right-derivatives h'_{t+} as well as non-increasing left-continuous left-derivatives

[7] Of course all z_i together have to sum up to $(T-t+1)z^*$ and not all z_i can be equal to z^*. These however are the only restrictions.

[8] Any such sequence yields a weighted arithmetic mean that lies within N_{z^o}.

[9] I have to show that concavity does not only hold for convex combinations within a particular set N_{z^o} but for all convex combinations within Γ_t.

h'_{t-} at every point in N_{z^o} (van Tiel 1984, 4,5). Moreover there are at most countably many points in N_{z^o} where h_t is not differentiable (van Tiel 1984, 5). Take any closed interval $[z^l, z^u] \subset \Gamma_t$. Then already a finite number of open sets N_{z^o} with $z^o \in I \subseteq \Gamma_t$, I finite, cover $[z^l, z^u]$ (Heine-Borel-theorem). Hence there are just countably many points where h_t is not differentiable on $[z^l, z^u]$. Denote the countable set where h_t is not differentiable by A. Then on $[z^l, z^u]\backslash A$ it is $h'_{t-} = h'_{t+}$ and due to the left-continuity of the left-derivative and right-continuity of the right-derivative $h_t{}'$ is continuous on $[z^l, z^u]\backslash A$. Moreover for all points in A the left- and right-derivative exist. But for such an almost everywhere continuously differentiable function the fundamental theorem of calculus applies (Königsberger 1995, 217). Therefore the relation $h_t(z) = h_t(c) + \int_c^z h'_{t+}(z')\, dz$, $c, z \in [z^l, z^u]$ holds. By van Tiel (1984, 9) such an integral representation with a right-continuous non-increasing integrand is a sufficient condition for weak concavity of h_t on $[z^l, z^u]$. Moreover any open set $\Gamma_t \subset \mathbb{R}$ is exhaustible by compact sets, i.e there exists an isotone sequence of closed intervals $[z^l_n, z^u_n]_{,n\in\mathbb{N}}$ such that $\Gamma_t = \bigcup_{n\in\mathbb{N}}[z^l_n, z^u_n]$. Hence h_t has to be weakly concave on Γ_t.

Second, I show that local strict concavity together with global weak concavity implies strict concavity on Γ_t. Take any pair of points $z_a, z_b \in \Gamma_t, z_a < z_b$. Let $z_c \in N_{z_b}$ be a point satisfying $z_a < z_c < z_b$. Moreover define $\lambda \in (0,1)$ by $z_c = \lambda z_a + (1-\lambda) z_b$ and let $\mu = \frac{1}{2\lambda}$. Then the following inequality holds for any pair $z_a \neq z_b$ in Γ_t (as $z_a < z_b$ is wlog):

$$\begin{aligned}
f_t g_t^{-1}\big(\tfrac{1}{2} z_a + \tfrac{1}{2} z_b\big) &= f_t g_t^{-1}\big(\mu\,\lambda\, z_a + (1-\mu\,\lambda)\, z_b\big) \\
&= f_t g_t^{-1}\big(\mu\,\lambda\, z_a + \big(\mu\,(1-\lambda) + (1-\mu)\big)\, z_b\big) \\
&= f_t g_t^{-1}\big(\mu\,\underbrace{(\lambda\, z_a + (1-\lambda)\, z_b)}_{z_c} + (1-\mu)\, z_b\big) \\
&> \mu\; f_t g_t^{-1}\big(\lambda z_a + (1-\lambda)\, z_b\big) + (1-\mu)\; f_t g_t^{-1}(z_b) \\
&\geq \mu\; \big(\lambda\, f_t g_t^{-1}(z_a) + (1-\lambda)\, f_t g_t^{-1}(z_b)\big) + (1-\mu)\; f_t g_t^{-1}(z_b) \\
&= \mu\,\lambda\, f_t g_t^{-1}(z_a) + \big(\mu\,(1-\lambda) + (1-\mu)\big)\, f_t g_t^{-1}(z_b) \\
&= \mu\,\lambda\, f_t g_t^{-1}(z_a) + (1-\mu\lambda)\, f_t g_t^{-1}(z_b) \\
&= \tfrac{1}{2} f_t g_t^{-1}(z_a) + \tfrac{1}{2}\, f_t g_t^{-1}(z_b)\,.
\end{aligned}$$

Therefore $f_t g_t^{-1}$ is strictly concave on Γ_t (Hardy et al. 1964, 75).

Part IV ("⇐"): It is left to proof that strict concavity on Γ_t implies axiom A6$^{\text{s}}$. As in part one of this proof the prerequisite of A6^{s} becomes

$$\begin{array}{rcl} \mathrm{x}^t & \sim_t & \mathrm{x}'^t \\ \Rightarrow g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}_\tau^t) + \xi_t\Big] & = & g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}'^t_\tau) + \xi_t\Big]. \end{array} \tag{C.17}$$

The existence of $i \in \{t, ..., T\}$ such that $[\mathrm{x}_i^t]_i \not\sim_i [\mathrm{x}'^t_i]_i$ translates into

$$\begin{array}{rcl} g_\tau u_\tau(\mathrm{x}_i^t) & \neq & g_\tau u(\mathrm{x}'^t_i) \\ \Leftrightarrow \theta_t \sum_{\substack{\tau=t \\ \tau\neq i}}^{T} g_\tau u(\mathrm{x}_\tau^t) + \theta_t g_i u_\tau(\mathrm{x}_i^t) + \xi_t & \neq & \sum_{\substack{\tau=t \\ \tau\neq i}}^{T} g_\tau u(\mathrm{x}_\tau^t) + \theta_t g_\tau u(\mathrm{x}'^t_i) + \xi_t \\ \Leftrightarrow \qquad \tilde{z}(\mathrm{x}^t) & \neq & \tilde{z}\big((\mathrm{x}_{-i}^t \mathrm{x}'^t_i)\big) \end{array} \tag{C.18}$$

for some $i \in \{t, ...T\}$. But then due to strict concavity of $f_t \circ g_t^{-1}$, the fact that $\tilde{z}\big((\mathrm{x}_{-i}^t \mathrm{x}'^t_i)\big)$ cannot be the same for all i,[10] and using equation (C.17) it has to hold that

$$\begin{aligned} & f_t g_t^{-1}\left[\sum_{i=t}^{T} \tfrac{1}{T-t+1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\left((\mathrm{x}_{-i}^t \mathrm{x}'^t_i)_\tau\right) + \xi_t\right]\right] > \\ & \qquad \sum_{i=t}^{T} \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\left((\mathrm{x}_{-i}^t \mathrm{x}'^t_i)_\tau\right) + \xi_t\right] \\ \Rightarrow\ & f_t g_t^{-1}\left[\tfrac{T-t}{T-t+1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}_\tau^t) + \xi_t\right] + \tfrac{1}{T-t+1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}'^t_\tau) + \xi_t\right]\right] > \\ & \qquad \sum_{i=t}^{T} \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\left((\mathrm{x}_{-i}^t \mathrm{x}'^t_i)_\tau\right) + \xi_t\right] \\ \Rightarrow\ & g_t^{-1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}_\tau^t) + \xi_t\right] > \\ & \qquad f_t^{-1}\left[\sum_{i=t}^{T} \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau\left((\mathrm{x}_{-i}^t \mathrm{x}'^t_i)_\tau\right) + \xi_t\right]\right] \\ \Rightarrow\ & \quad \mathrm{x}^t \quad \succ_T \quad \sum_{i=t}^{T} \tfrac{1}{T-t+1}\left(\mathrm{x}_{-i}^t \mathrm{x}'^t_i\right). \end{aligned}$$

Note that the flow of manipulations is laid out in more detail (going backwards) in part two of the proof.

Part V: Assertion b) is obtained by replacing A6^{s} by A6^{w}and the strict inequaties by their weak counterparts.[11] A decision maker is intertemporally risk neutral if his preferences satisfy weak risk seeking as well as weak risk aversion. Therefore, assertion b) implies that the function $f_t \circ g_t^{-1}$ has to be concave and convex at the same time and, thus, linear. On the other hand, a representation featuring a linear composition $f_t \circ g_t^{-1}$ yields indifference between the certain consumption path and the lottery and, therefore, satisfies weak risk seeking as well as weak risk aversion (compare part four of the proof). In consequence, assertion c) holds. The proof of assertion d) is completely

[10] This is implied by equation (C.18) as again $\tilde{z}(\mathrm{x}^t)$ equals the weighted average $\frac{1}{T-t+1}\sum_{i=t}^{T} \tilde{z}\big((\mathrm{x}_{-i}^t \mathrm{x}'^t_i)\big)$.

[11] In this case the second step in part three becomes redundant.

analogous to that of assertion a). Equation (C.15) becomes

$$f_t g_t^{-1}\left(\tfrac{1}{2}\, z^{\text{high}} + \tfrac{1}{2}\, z^{\text{low}}\right) \quad > \tfrac{1}{2} f_t g_t^{-1}(z^{\text{high}}) + \tfrac{1}{2} f_t g_t^{-1}(z^{\text{low}})\,,$$

implying that the last step ("Finally...") in part three of the proof can be omitted. □

Proof of lemma 5: Let the triples $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ and $(\tilde{u}_t, \tilde{f}_t, \tilde{g}_t)_{t\in\{1,...,T\}}$ be arbitrary representations for the set of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 4. For every $t \in \{1, ..., T\}$ there exist, as in the proof of lemma 3, strictly increasing continuous function s_t and affine transformations $\mathfrak{a}_t^+ \in \mathsf{A}^+$, as well as $a \in R_{++}$ and affine transformations $\mathfrak{a}_t^a \in \mathsf{A}^a$, such that $\tilde{u}_t = s_t \circ u_t$, $\tilde{f}_t = \mathfrak{a}_t^+ f_t \circ s_t^{-1}$ and $\tilde{g}_t = \mathfrak{a}_t^a\, g_t \circ s_t^{-1}$.
a) The requirement that for some t^* both representations satisfy the condition $\Delta G_{t*} = \overline{G}_{t^*} - \underline{G}_{t^*} = g_{t^*} \circ u_{t^*}(x^{\max}) - g_{t^*} \circ u_{t^*}(x^{\min}) \stackrel{!}{=} \mathcal{W}^*$ implies

$$\begin{aligned}
\mathcal{W}^* &= \overline{\tilde{G}}_{t^*} - \underline{\tilde{G}}_{t^*} = \tilde{g}_{t^*} \circ \tilde{u}_{t^*}(x^{\max}) - \tilde{g}_{t^*} \circ \tilde{u}_{t^*}(x^{\min}) \\
&= \mathfrak{a}_{t^*}^a\, g_{t^*} \circ s_{t^*}^{-1} \circ \tilde{u}_{t^*}(x^{\max}) - \mathfrak{a}_{t^*}^a\, g_{t^*} \circ s_{t^*}^{-1} \circ \tilde{u}_{t^*}(x^{\min}) \\
&= \mathfrak{a}_{t^*}^a\, g_{t^*} \circ u_{t^*}(x^{\max}) - \mathfrak{a}_{t^*}^a\, g_{t^*} \circ u_{t^*}(x^{\min}) \\
&= a g_{t^*} \circ u_{t^*}(x^{\max}) + b_{t^*} - a g_{t^*} \circ u_{t^*}(x^{\min}) - b_{t^*} \\
&= a\left(\overline{G}_{t^*} - \underline{G}_{t^*}\right) = a\mathcal{W}^*\,.
\end{aligned}$$

Therefore, $a = 1$ and, as the multiplicative constant is the same for all periods, the remaining freedom of the expression $f_t \circ g_t^{-1}$ corresponds to transformations $f_t \circ g_t^{-1} \rightarrow \tilde{f}_t \circ \tilde{g}_t^{-1} = \mathfrak{a}_t^+ f_t \circ g_t^{-1} \mathfrak{a}_t^{1^{-1}}$, where $\mathfrak{a}_t^{1^{-1}}$ denotes the inverse of $\mathfrak{a}_t^{a=1}$, i.e. $\mathfrak{a}_t^{1^{-1}}(z) = z - b_t$.

To compare the functions $f_t \circ g_t^{-1}$ and $\tilde{f}_t \circ \tilde{g}_t^{-1}$ characterizing intertemporal risk aversion in the different representations, I first have to work out how the argument

$$\begin{aligned}
z &= \theta_t g_t \circ u_t(x_t) + \theta_t \theta_{t+1}^{-1} g_{t+1}\left(\mathcal{M}^f(p_{t+1}, u_{t+1}\right) + \theta_t \theta_{t+1}^{-1} \vartheta_t \\
&= \theta_t g_t u_t(x_t) + (1 - \theta_t) \tfrac{\Delta G_t}{\Delta G_{t+1}} \left(g_{t+1} \mathcal{M}^f(p_{t+1}, u_{t+1}) + \vartheta_t\right)
\end{aligned}$$

scales under the above transformations. This step is necessary in order to compare both measures for the same consumption plans. To this end, first note that $a = 1$ implies $\Delta\tilde{G}_t = \Delta G_t$. Therefore it is $\tilde{\theta}_t = \theta_t$. Moreover, simultaneous transformations $(u_t, g_t) \rightarrow (\tilde{u}, \tilde{g}_t) = (s_t u_t, g_t s_t^{-1})$ leave the normalization constant ϑ_t unchanged, because $\overline{G}_t = g_t \circ u_t(x_t^{\max}) = g_t s_t^{-1} s_t g_t(x_t^{\max}) = \tilde{g}_t \circ \tilde{u}_t(x_t^{\max}) = \overline{\tilde{G}}_t$ and similarly $\underline{G}_t = \underline{\tilde{G}}_t$. Therefore, by equation (C.4) it follows that general transformations $(u_t, f_t, g_t) \rightarrow (\tilde{u}_t, \tilde{f}_t, \tilde{g}_t) =$

$(s_t u_t, \mathfrak{a}_t^+ f_t s_t^{-1}, g_t s_t^{-1} + b_t)$ yield $\tilde{\vartheta}_t = \vartheta_t - b_{t+1} + b_t \frac{\Delta G_{t+1}}{\Delta G_t}$. With these results, I find

$$\begin{aligned}
\tilde{z} &= \tilde{\theta}_t \tilde{g}_t \tilde{u}_t(x_t) + (1-\tilde{\theta}_t)\tfrac{\Delta \tilde{G}_t}{\Delta \tilde{G}_{t+1}} \left(\tilde{g}_{t+1}\mathcal{M}^f(p_{t+1}, \tilde{u}_{t+1}) + \tilde{\vartheta}_t\right) \\
&= \theta_t \left(g_t s_t^{-1} s_t u_t(x_t) + b_t\right) + \\
&\quad (1-\theta_t)\tfrac{\Delta G_t}{\Delta G_{t+1}} \left(g_{t+1} s_{t+1}^{-1} s_{t+1}\mathcal{M}^f(p_{t+1}, u_{t+1}) + b_{t+1} + \vartheta_t - b_{t+1} + b_t \tfrac{\Delta G_{t+1}}{\Delta G_t}\right) \\
&= \theta_t g_t u_t(x_t) + (1-\theta_t)\tfrac{\Delta G_t}{\Delta G_{t+1}} \left(g_{t+1}\mathcal{M}^f(p_{t+1}, u_{t+1}) + \vartheta_t\right) + \\
&\quad +\theta_t b_t + (1-\theta_t)\tfrac{\Delta G_t}{\Delta G_{t+1}} \left(b_t \tfrac{\Delta G_{t+1}}{\Delta G_t}\right) \\
&= \theta_t g_t u_t(x_t) + (1-\theta_t)\tfrac{\Delta G_t}{\Delta G_{t+1}} \left(g_{t+1}\mathcal{M}^f(p_{t+1}, u_{t+1}) + \vartheta_t\right) + b_t \\
&= z + b_t .
\end{aligned}$$

In consequence, for twice differentiable functions $f_t \circ g_t$, it follows by equation (7.8) that

$$\widetilde{\mathrm{AIRA}}_t(\tilde{z})\Big|_{\tilde{z}=z+b_t} = -\frac{\left(\tilde{f}_t \circ \tilde{g}_t^{-1}\right)''(\tilde{z})}{\left(\tilde{f}_t \circ \tilde{g}_t^{-1}\right)'(\tilde{z})}\Bigg|_{\tilde{z}=z+b_t} = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)} = \mathrm{AIRA}_t(z)\,.$$

Thus, the measures of absolute intertemporal risk aversion AIRA_t are independent of the particular choice of the triples $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ representing the underlying preferences $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$.

b) The requirement that $g_t \circ u_t(x_t^{\mathrm{zero}}) = \tilde{g}_t \circ \tilde{u}_t(x_t^{\mathrm{zero}}) = 0$ for all $t \in \{1, ..., T\}$ yields

$$\begin{aligned}
0 &= \tilde{g}_t \circ \tilde{u}_t(x_t^{\mathrm{zero}}) \\
&= a g_t \circ s_t^{-1} s_t u_t(x_t^{\mathrm{zero}}) + b_t \\
&= a g_t \circ u_t(x_t^{\mathrm{zero}}) + b_t \\
&= a \cdot 0 + b_t = b_t\,.
\end{aligned}$$

Therefore it is $\tilde{f}_t \circ \tilde{g}_t^{-1} = \mathfrak{a}^+ f_t g_t^{-1} a^{-1}$. To compare the functions $f_t \circ g_t^{-1}$ and $\tilde{f}_t \circ \tilde{g}_t^{-1}$, characterizing intertemporal risk aversion, I have to relate the arguments z and $\tilde{z}$ corresponding to the same consumption plan. As in the proof of assertion a), it holds that the transformations s_t leave $\overline{G}_t$, $\underline{G}_t$, and ΔG_t unchanged. Therefore, I know for the general transformations $(u_t, f_t, g_t) \to (\tilde{u}_t, \tilde{f}_t, \tilde{g}_t) = (s_t u_t, \mathfrak{a}_t^+ f_t s_t^{-1}, a g_t s_t^{-1})$ by equation (C.3) that $\tilde{\theta}_t = \theta_t$ and by equation (C.4) that $\tilde{\vartheta} = a\vartheta_t$. With these results, I find

$$\begin{aligned}
\tilde{z} &= \tilde{\theta}_t \tilde{g}_t \tilde{u}_t(x_t) + (1-\tilde{\theta}_t)\tfrac{\Delta \tilde{G}_t}{\Delta \tilde{G}_{t+1}} \left(\tilde{g}_{t+1}\mathcal{M}^f(p_{t+1}, \tilde{u}_{t+1}) + \tilde{\vartheta}_t\right) \\
&= \theta_t a g_t s_t^{-1} s_t u_t(x_t) + (1-\theta_t)\tfrac{\Delta G_t}{\Delta G_{t+1}} \left(a g_{t+1} s_{t+1}^{-1} s_{t+1}\mathcal{M}^f(p_{t+1}, u_{t+1}) + a\vartheta_t\right) \\
&= a\left(\theta_t g_t u_t(x_t) + (1-\theta_t)\tfrac{\Delta G_t}{\Delta G_{t+1}} \left(g_{t+1}\mathcal{M}^f(p_{t+1}, u_{t+1}) + \vartheta_t\right)\right) \\
&= az.
\end{aligned}$$

In consequence, for twice differentiable functions $f_t \circ g_t$, it follows by equation (7.6) that

$$\widetilde{\mathrm{RIRA}}_t(\tilde{z})\Big|_{\tilde{z}=az} = -\frac{\left(\tilde{f}_t \circ \tilde{g}_t^{-1}\right)''(\tilde{z})}{\left(\tilde{f}_t \circ \tilde{g}_t^{-1}\right)'(\tilde{z})}\,\tilde{z}\Bigg|_{\tilde{z}=az} = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)}\, z = \mathrm{RIRA}_t(z)\,.$$

Thus, the measures of relative intertemporal risk aversion RIRA_t are independent of the particular choice of the triples $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ representing the underlying preferences $\succeq= (\succeq_t)_{t\in\{1,...,T\}}$.
c) Simply combine statements a) and b), acknowledging that the requirements are disjoint in the sense that a) fixes the multiplicative parameter a, while b) fixes the translational parameters b_t.
d) As in the proof of asserion a), it follows that $a = 1$. Therefore, the requirement $\underline{G}_t = \underline{\tilde{G}}_t = w_t$ for all $t \in \{1, ..., T\}$ implies

$$\begin{aligned} w_t &= \underline{\tilde{G}}_t = \tilde{g}_t\tilde{u}_t(x^{\min}) \\ &= g_t s_t^{-1} s_t u_t(x^{\min}) + b_t = g_t u_t(x^{\min}) + b_t \\ &= w_t + b_t, \end{aligned}$$

and thus $b_t = 0$ for all $t \in \{1, ..., T\}$. But then, the rest of the proof is equivalent to that of assertions a) and b). □

C.2 Proofs for Chapter 9

Proof of theorem 7: The proof is divided into four parts. Axioms A1-A3, A4' and A5' assure the existence of a representation in the sense of theorem 4. In the first part I show that axiom A7 allows to pick the same Bernoulli utility for all periods. In the second part I work out a relation between the functions g_t in different periods that has to hold in such a representation by axiomA7. Part three calculates the corresponding normalization constants and brings about the representation stated in the theorem. Finally, part four proves the necessity of the axioms.
Part I (“⇒”): I show that axiom A7 implies that there exists a strictly monotonic and continuous transformation s_t such $u_{t-1} = s_t \circ u_t$ for any $t \in \{1, .., T\}$. To this end,

translate axiom A7 into the representation of theorem 4 using equation (C.1).

$$
\begin{aligned}
&(\mathsf{x}^2, x^0) \quad \succeq_1 \quad (\mathsf{x}'^2, x^0) \qquad \Leftrightarrow \qquad \mathsf{x}^2 \quad \succeq_2 \mathsf{x}'^2 \\
\rightsquigarrow\ &\tilde{u}_1\big((\mathsf{x}^2, x^0)\big) \ \geq\ \tilde{u}_1\big((\mathsf{x}'^2, x^0)\big) \quad \Leftrightarrow \quad \tilde{u}_1(\mathsf{x}^2) \ \geq\ \tilde{u}_1(\mathsf{x}'^2) \\
\rightsquigarrow\ &g_1^{-1}\left(\theta_1 \sum_{\tau=2}^{T} g_{\tau-1} \circ u_{\tau-1}(\mathsf{x}^2_\tau) + g_T \circ u_T(x^0) + \xi_1\right) \\
&\geq g_1^{-1}\left(\theta_1 \sum_{\tau=2}^{T} g_{\tau-1} \circ u_{\tau-1}(\mathsf{x}'^2_{\tau-1}) + g_T \circ u_T(x^0) + \xi_1\right) \\
\Leftrightarrow\ &g_2^{-1}\left(\theta_2 \sum_{\tau=2}^{T} g_\tau \circ u_\tau(\mathsf{x}^2_\tau) + \xi_2\right) \geq g_2^{-1}\left(\theta_2 \sum_{\tau=2}^{T} g_\tau \circ u_\tau(\mathsf{x}'^2_\tau) + \xi_2\right)
\end{aligned}
$$

Considering in particular the consumption paths $\mathsf{x}^2, \mathsf{x}'^2$ satisfying $\mathsf{x}^2_\tau = \mathsf{x}'^2_\tau \, \forall \, \tau \neq t$ yields

$$
\begin{aligned}
\rightsquigarrow\ & g_{t-1} \circ u_{t-1}(\mathsf{x}^2_t) \geq g_{t-1} \circ u_{t-1}(\mathsf{x}'^2_t) \Leftrightarrow g_t \circ u_t(\mathsf{x}^2_t) \geq g_t \circ u_t(\mathsf{x}'^2_t) \\
\rightsquigarrow\ & \quad u_{t-1}(\mathsf{x}^2_t) \quad \geq \quad u_{t-1}(\mathsf{x}'^2_t) \quad \Leftrightarrow \quad u_t(\mathsf{x}^2_t) \quad \geq u_t(\mathsf{x}'^2_t)
\end{aligned}
$$

for all $\mathsf{x}^2_t = \mathsf{x}'^2_t \in X$. Therefore, as in the proof of proposition 7, it has to exist a strictly monotonic and continuous transformation s_t such that $u_{t-1} = s_t \circ u_t$. But then, by induction it is $B_{\succeq_1} = B_{\succeq_2} = ... = B_{\succeq_T} \equiv B_{\succeq}$ and I can pick a common Bernoulli utility function $u \in B_{\succeq}$ for all periods.

Part II ("⇒"): In this part, I derive an affine relation between the functions g_t in different periods. To this end, I translate axiom A7 into the particular representation in the sense of theorem 4, which applies the same Bernoulli utility function u for all periods. Using again equation (C.1) I obtain the condition

$$
\begin{aligned}
&\sum_{\tau=2}^{T} g_{\tau-1} \circ u_{\tau-1}(\mathsf{x}^2_\tau) + \cancel{g_T \circ u_T(x^0)} \geq \sum_{\tau=2}^{T} g_{\tau-1} \circ u_{\tau-1}(\mathsf{x}'^2_{\tau-1}) + \cancel{g_T \circ u_T(x^0)} \\
\Leftrightarrow &\sum_{\tau=2}^{T} g_\tau \circ u_\tau(\mathsf{x}^2_\tau) \qquad\qquad\qquad\quad \geq \sum_{\tau=2}^{T} g_\tau \circ u_\tau(\mathsf{x}'^2_\tau)
\end{aligned}
$$

for all $\mathsf{x}^2, \mathsf{x}'^2 \in \mathsf{X}^2$. The above equivalence implies that both, $\sum_{\tau=2}^{T} g_\tau \circ u_\tau(\mathsf{x}^2_\tau)$ and $\sum_{\tau=2}^{T} g_{\tau-1} \circ u_{\tau-1}(\mathsf{x}^2_\tau)$, are representations for $\succeq_2 |_{\mathsf{X}^2}$. In consequence, by the moreover part of theorem 4 there exist $a \in \mathbb{R}_{++}$ and $b_t \in \mathbb{R}, t \in \{1, ..., T-1\}$, such that $g_t = a g_{t+1} + b_t$ for all $t \in \{1, ..., T-1\}$.[12] Use the freedom in the uniqueness of $(g_t)_{t \in \{1,...,T\}}$ to define $\tilde{g}_t = g_t - \sum_{\tau=t}^{T-1} a^{\tau-t} b_\tau$ for $t \in \{1, ...T-1\}$ without loosing the rep-

[12]Here it is $g'_t = g_{t+1}$. Note that it is immediate from the proof of the moreover part in theorem 4 that coincidence of the representations (only) on the certain outcome paths is enough to assure the uniqueness result for $(g_t)_t \in \{1, ..., T\}$.

resentative character of the sequence $(u, f_t, \tilde{g}_t)_{t\in\{1,...,T\}}$ for $(\succeq_t)_{t\in\{1,...,T\}}$. Observe that $\tilde{g}_t = g_t - \sum_{\tau=t}^{T-1} a^{\tau-t} b_\tau = a g_{t+1} + b_t - b_t - a \sum_{\tau=t+1}^{T-1} a^{\tau-t} b_\tau = a\tilde{g}_{t+1}$. Set $g = a^{T-1}\tilde{g}_T$. Moreover let $\beta = a^{-1}$. Then the sequence of triples $(u, f_t, a^{T-t}\tilde{g}_T) = (u, f_t, \beta^{t-T}\beta^{T-1}g) = (u, f_t, \beta^{t-1}g)$ for $t \in \{1, ..., T\}$ represents $(\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 4.

Note 1: Expressing the triples with respect to $\tilde{g}_\tau$ instead of g yields the equivalent representation triples $(u, f_t, \beta^{t-\tau}\tilde{g}_\tau)_{t\in\{1,...,T\}}$ and in particular for $\tau = T$ the representation $(u, f_t, \beta^{t-T}\tilde{g}_T)_{t\in\{1,...,T\}}$.

Part III ("⇒"): Calculating the corresponding normalization constants for the representing tuples derived in the previous step, yields the representation stated in the theorem. In the usual convention denote $\Delta G_t = \overline{G}_t - \underline{G}_t$ and $G = [\underline{G}, \overline{G}] = [g(\min_{x\in X} u(x)), g(\max_{x\in X} u(x))]$ and find

$$\begin{aligned}
\theta_t &= \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau} = \frac{\beta^t \Delta G}{\sum_{\tau=t}^T \beta^\tau \Delta G} = \frac{1}{1+\beta+\beta^2+...+\beta^{T-t}} = \frac{1-\beta}{1-\beta^{T-t+1}} && \text{for } \beta \neq 1,\\
\theta_t &= \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau} = \frac{\Delta G}{\sum_{\tau=t}^T \Delta G} = \frac{\Delta G}{(T-t+1)\Delta G} = \frac{1}{T-t+1} && \text{for } \beta = 1 \text{ and}\\
\vartheta_t &= \frac{\overline{G}_{t+1}\underline{G}_t - \underline{G}_{t+1}\overline{G}_t}{\Delta G_t} = \frac{\beta^{t+1}\overline{G}\,\beta^t\underline{G} - \beta^{t+1}\underline{G}\,\beta^t\overline{G}}{\beta^t \Delta G} = 0.
\end{aligned}$$

Using equation (C.2) it is straight forward to calculate the aggregate intertemporal utility functions. In the case $\beta \neq 1$ they are

$$\begin{aligned}
\tilde{u}_t(\cdot,\cdot) &= \tilde{g}_t^{-1}\Big[\theta_t \tilde{g}_t \circ u(\cdot) \ + (1-\theta_t)\frac{\Delta\tilde{G}_t}{\Delta\tilde{G}_{t+1}}\Big(\tilde{g}_{t+1} \circ \mathcal{M}^{f_{t+1}}(\cdot, \tilde{u}_{t+1}) + \ 0\ \Big)\Big\}\Big]\\
&= g^{-1}\Big[\beta^{-t+1}\Big\{\ \theta_t \beta^{t-1} g \circ u(\cdot) \ + (1-\theta_t)\beta^{-1}\Big(\beta^t g \circ \mathcal{M}^{f_{t+1}}(\cdot, \tilde{u}_{t+1})\Big)\Big\}\Big]\\
&= g^{-1}\Big[\ \theta_t g \circ u(\cdot) \ + (1-\theta_t) g \circ \mathcal{M}^{f_{t+1}}(\cdot, \tilde{u}_{t+1})\Big].
\end{aligned}$$

Defining $\beta_t = 1 - \theta_t = 1 - \frac{1-\beta}{1-\beta^{T-t+1}} = \frac{1-\beta^{T-t+1}-1+\beta}{1-\beta^{T-t+1}} = \beta\frac{1-\beta^{T-t}}{1-\beta^{T-t+1}}$ gives the representation stated in the theorem. For $\beta = 1$ find

$$\tilde{u}_t(\cdot,\cdot) = g^{-1}\Big[\frac{1}{T-t+1} g \circ u(\cdot) \ + (1 - \frac{1}{T-t+1}) g \circ \mathcal{M}^{f_{t+1}}(\cdot, \tilde{u}_{t+1})\Big]$$

and define $\beta_t = 1 - \theta_t = 1 - \frac{1}{T-t+1} = \frac{T-t+1-1}{T-t+1} = \frac{T-t}{T-t+1}$ to get the stated representation.

Note 2: For the evaluation of certain consumption paths equation (C.1) together with $\vartheta_t = 0$ and hence $\xi_t = 0$ yields:

$$\tilde{u}_t(\mathrm{x}^t) = \tilde{g}_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} \tilde{g}_\tau \circ u_\tau(\mathrm{x}_\tau^t)\Big] = \ g^{-1}\Big[\ (1-\beta_t) \sum_{\tau=t}^{T} \beta^{\tau-t} g \circ u(\mathrm{x}_\tau^t)\ \Big]. \tag{C.19}$$

Part IV ("⇐"): Axioms A1-A5 follow immediately from "⇐" of theorem 4. To see that axiom A7 holds, take a look at equation (C.19) and note that g^{-1} and the x^0 term cancel in the representation of A7 (for any x^0).

Moreover part: The moreover part is an immediate consequence of the moreover part of theorem 4. □

Proof of theorem 8: "⇒": Axioms A1-A7 assure the existence of a representation in the sense of theorem 7. Axiom A8 implies furthermore that the uncertainty aggregation rules in different periods can be characterized by the same function. Using equation (C.19) to translate axiom A8 into the representation of theorem 7 yields for the first expression

$$\begin{array}{lcc}
\frac{1}{2}\bar{\mathrm{x}}^{t} + \frac{1}{2}\bar{\mathrm{x}}'^{\,t} & \succeq_t & \bar{\mathrm{x}}''^{\,t} \\
\Leftrightarrow \quad f_t^{-1}\left[\frac{1}{2}f_t \circ \tilde{u}_t(\bar{\mathrm{x}}^{t}) + \frac{1}{2}f_t \circ \tilde{u}_t(\bar{\mathrm{x}}'^{\,t})\right] & \geq & \tilde{u}_t(\bar{\mathrm{x}}''^{\,t}) \\
\Leftrightarrow f_t^{-1}\Big[\frac{1}{2}f_t \circ g^{-1}\big[(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}g\circ u(\bar{x})\big] + \frac{1}{2}f_t \circ g^{-1}\big[(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}g\circ u(\bar{x}')\big]\Big] & & \\
 & \geq & g^{-1}\big[\,(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}g\circ u(\bar{x}'')\,\big] \\
\Leftrightarrow \; f_t^{-1}\Big[\frac{1}{2}f_t \circ g^{-1}\big[\frac{1-\beta}{1-\beta^{T-t+1}}\,\frac{1-\beta^{T-t+1}}{1-\beta}\,g\circ u(\bar{x})\big] + \frac{1}{2}f_t \circ g^{-1}\big[\frac{1-\beta}{1-\beta^{T-t+1}}\,\frac{1-\beta^{T-t+1}}{1-\beta}\,g\circ u(\bar{x}')\big]\Big] & & \\
 & \geq & g^{-1}\big[\,\frac{1-\beta}{1-\beta^{T-t+1}}\,\frac{1-\beta^{T-t+1}}{1-\beta}\,g\circ u(\bar{x}'')\,\big] \\
\Leftrightarrow \quad f_t^{-1}\Big[\frac{1}{2}f_t\circ u(\bar{x}) + \frac{1}{2}f_t\circ u(\bar{x}')\Big] & \geq & u(\bar{x}'')\,,
\end{array}$$

and analogously for the second expression

$$\begin{array}{lcc}
\frac{1}{2}\bar{\mathrm{x}}^{t+1} + \frac{1}{2}\bar{\mathrm{x}}'^{\,t+1} & \succeq_{t+1} & \bar{\mathrm{x}}''^{\,t+1} \\
\Leftrightarrow f_{t+1}^{-1}\left[\frac{1}{2}f_{t+1}\circ\tilde{u}_{t+1}(\bar{\mathrm{x}}^{t+1}) + \frac{1}{2}f_{t+1}\circ\tilde{u}_{t+1}(\bar{\mathrm{x}}'^{\,t+1})\right] & \geq & \tilde{u}_{t+1}(\bar{\mathrm{x}}''^{\,t+1}) \\
\Leftrightarrow \quad f_{t+1}^{-1}\Big[\frac{1}{2}f_{t+1}\circ u(\bar{x}) + \frac{1}{2}f_{t+1}\circ u(\bar{x}')\Big] & \geq & u(\bar{x}'')\,.
\end{array}$$

For all $\bar{x}, \bar{x}' \in X$ there is an outcome $\bar{x}'' \in X$ such that the above relations hold with equality (compare proof of theorem 2). This fact implies that the following equality has to hold for all $\bar{x}, \bar{x}' \in X$:

$$\begin{array}{rcl}
f_t^{-1}\Big[\frac{1}{2}f_t\circ u(\bar{x}) + \frac{1}{2}f_t\circ u(\bar{x}')\Big] & = & f_{t+1}^{-1}\Big[\frac{1}{2}f_{t+1}\circ u(\bar{x}) + \frac{1}{2}f_{t+1}\circ u(\bar{x}')\Big] \\
\Leftrightarrow \; f_{t+1}f_t^{-1}\Big[\frac{1}{2}f_t u(\bar{x}) + \frac{1}{2}f_t u(\bar{x}')\Big] & = & \frac{1}{2}f_{t+1}\,f_t^{-1}f_t\,u(\bar{x}) + \frac{1}{2}f_{t+1}\,f_t^{-1}f_t\,u(\bar{x}')\,.
\end{array}$$

Defining $h_t = f_{t+1}\circ f_t^{-1}$ and the interval $\mathsf{F}_t = f_t(U)$, this condition translates into the equation

$$h_t\left(\tfrac{1}{2}y + \tfrac{1}{2}y'\right) = \tfrac{1}{2}h_t\,(y) + \tfrac{1}{2}h_t\,(y') \qquad \forall y, y' \in \mathsf{F}_t\,.$$

Therefore h_t has to be linear on F_t (Hardy et al. 1964, refinement of theorem 83 on p.74). Hence the expression $f_{t+1}\circ f_t^{-1}$ is linear on $f_t(U)$ implying that there exists $\mathfrak{a}_t \in \mathsf{A}$ such that with $z = f_t^{-1}(y) \in U$ it is

$$\begin{array}{rcl}
f_{t+1}f_t^{-1}(y) & = & \mathfrak{a}_t^{-1}\,y \\
\Leftrightarrow \quad f_t^{-1}(y) & = & f_{t+1}^{-1}\mathfrak{a}_t^{-1}\,y \\
\Leftrightarrow \quad f_t(z) & = & \mathfrak{a}_t f_{t+1}(z)\,.
\end{array}$$

By the fact that f_t and f_{t+1} are both increasing it follows that $\mathfrak{a}_t$ has to be *positive*

affine, i.e. $\mathfrak{a}_t \in \mathsf{A}^+$. But as each f_t in the representation is determined only up to positive affine transformations, setting $f_t = f_{t+1} = f$ still yields a representation of $\succeq_{t\in\{1,...,T\}}$.
"$\Leftarrow$": Axioms A1-A5 follow immediately from "$\Leftarrow$" of theorem 4. As seen in the first part of the proof, for constant consumption paths it is $\tilde{u}_t(\bar{\mathsf{x}}^t) = u(\bar{x}) \forall t \in \{1, ..., T\}$. Therefore axiom A8 is seen to hold by observing that for all $\bar{x}, \bar{x}', \bar{x}'' \in X$:

$$\begin{array}{rccc} & \frac{1}{2}\bar{\mathsf{x}}^{\,t} + \frac{1}{2}\bar{\mathsf{x}}'^{\,t} & \succeq_t & \bar{\mathsf{x}}''^{\,t} \\ \Leftrightarrow & f^{-1}\left[\frac{1}{2}f \circ \tilde{u}_t(\bar{\mathsf{x}}^{\,t}) + \frac{1}{2}f \circ \tilde{u}_t(\bar{\mathsf{x}}'^{\,t})\right] & \geq & \tilde{u}_t(\bar{\mathsf{x}}''^{\,t}) \\ \Leftrightarrow & f^{-1}\left[\frac{1}{2}f \circ \tilde{u}_{t+1}(\bar{\mathsf{x}}^{\,t+1}) + \frac{1}{2}f \circ \tilde{u}_{t+1}(\bar{\mathsf{x}}'^{\,t+1})\right] & \geq & \tilde{u}_{t+1}(\bar{\mathsf{x}}''^{\,t+1}) \\ \Leftrightarrow & \frac{1}{2}\bar{\mathsf{x}}^{\,t+1} + \frac{1}{2}\bar{\mathsf{x}}'^{\,t+1} & \succeq_{t+1} & \bar{\mathsf{x}}''^{\,t+1}\,. \end{array}$$

Moreover part: The moreover part is an immediate consequence of the moreover part of theorem 4. □

Proof of lemma 6: Except for admitting decreasing functions f and g, when changing "increasing" into "monotonic" in theorem 8, the statements are special cases of lemma 4, corollary 6 and corollary 7. The decreasing functions come in the same way as in the proofs for chapter 6.4, by noting that $\mathcal{M}^{\mathfrak{a}f} = \mathcal{M}^f$ and analogously $\mathcal{N}^{\mathfrak{a}g} = \mathcal{N}^g$ for all $\mathfrak{a} \in \mathsf{A}$. Therefore, if the triple (u, f, g) represents $\succeq_{t\in\{1,...,T\}}$ in the sense of theorem 8, then so do the triples $(u, -f, g)$ and $(u, f, -g)$, if f and g are admitted to be decreasing in the representation. □

Proof of theorem 9: The proof is divided into five parts. First, I translate axiom A9 into the representation of theorem 7. This step yields a requirement for the representing functions f_t and g that is solved in the second part under the assumption of differentiability of $f_t \circ g^{-1}$. The third part shows that the derived solution has to hold as well without assuming differentiability. Part four translates the solution into the representation stated in the theorem. Finally, part five proofs the necessity of the axioms for the representation.
Part I ("$\Rightarrow$"): First note that axiom A9 implies axiom A7 by choosing $\mathsf{x} = \mathsf{x}'$. Therefore a representation in terms of theorem 7 has to exist. In order to translate A9 for $t \in \{1, ..., T-1\}$ into the latter representation, note that by definition of x as an element of X^{t+1}, the period τ entry of the consumption path $(\mathsf{x}, x^0) \in \mathsf{X}^t$ corresponds to $(\mathsf{x}, x^0)_\tau = \mathsf{x}_{\tau+1}$ for $\tau \in \{t, ..., T-1\}$. Then, using equation (C.19), the left hand side of

the equivalence in axiom A9 translates into

$$\begin{aligned}
&\tfrac{1}{2}(\mathsf{x}, x^0) + \tfrac{1}{2}(\mathsf{x}', x^0) \succeq_t (\mathsf{x}'', x^0) \\
\Leftrightarrow\ & f_t^{-1}\Big\{ \tfrac{1}{2} f_t g^{-1}\Big[(1-\beta_t)\textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} gu(\mathsf{x}_{\tau+1}) + (1-\beta_t)\beta^{T-t} gu(x^0)\Big] \\
&\quad + \tfrac{1}{2} f_t g^{-1}\Big[(1-\beta_t)\textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} gu(\mathsf{x}'_{\tau+1}) + (1-\beta_t)\beta^{T-t} gu(x^0)\Big]\Big\} \\
&\geq g^{-1}\Big[(1-\beta_t)\textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} gu(\mathsf{x}''_{\tau+1}) + (1-\beta_t)\beta^{T-t} gu(x^0)\Big] \\
\Leftrightarrow\ & g f_t^{-1}\Big\{ \tfrac{1}{2} f_t g^{-1}\Big[(1-\beta_t)\textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} gu(\mathsf{x}_{\tau}) + (1-\beta_t)\beta^{T-t} gu(x^0)\Big] \\
&\quad + \tfrac{1}{2} f_t g^{-1}\Big[(1-\beta_t)\textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} gu(\mathsf{x}'_{\tau}) + (1-\beta_t)\beta^{T-t} gu(x^0)\Big]\Big\} \\
&\geq \Big[(1-\beta_t)\textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} gu(\mathsf{x}''_{\tau}) + (1-\beta_t)\beta^{T-t} gu(x^0)\Big]\,.
\end{aligned}$$

Define the sum $S = \sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} gu(\mathsf{x}_\tau)$ and similarly $S' = \sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} gu(\mathsf{x}'_\tau)$ and $S'' = \sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} gu(\mathsf{x}''_\tau)$ as well as $A = (1-\beta_t)\beta^{T-t} gu(x^0)$. Then, varying the consumption paths x, x' and x'' in X^{t+1}, goes along with varying S, S' and S'' in the interval $[\frac{1-\beta^{T-t}}{1-\beta}\,\underline{G}, \frac{1-\beta^{T-t}}{1-\beta}\,\overline{G}]$. Similarly, as x^0 is varied in X, the value A takes on any number in the interval $[(1-\beta_t)\beta^{T-t}\,\underline{G}, (1-\beta_t)\beta^{T-t}\,\overline{G}]$. In the introduced notation, the above inequality corresponding to the left hand side of the equivalence in axiom A9 writes as

$$g f_t^{-1}\Big\{ \tfrac{1}{2} f_t g^{-1}\Big[(1-\beta_t)S + A\Big] + \tfrac{1}{2} f_t g^{-1}\Big[(1-\beta_t)S' + A\Big]\Big\} - A \geq (1-\beta_t)S''. \tag{C.20}$$

In the same notation the right hand side of the equivalence in axiom A9 translates into

$$g f_{t+1}^{-1}\Big\{ \tfrac{1}{2} f_{t+1} g^{-1}\Big[(1-\beta_{t+1})S\Big] + \tfrac{1}{2} f_{t+1} g^{-1}\Big[(1-\beta_{t+1})S'\Big]\Big\} \geq (1-\beta_{t+1})S''. \tag{C.21}$$

As derived in the proof of theorem 4 (induction hypothesis H2), for every lottery $p_{t+1} \in P_{t+1}$ there exists a certain consumption path as certainty equivalent. In consequence, for any $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^{t+1}$, there exists a certainty equivalent $\mathsf{x}'' \in \mathsf{X}^{t+1}$ for the lottery $\frac{1}{2}\mathsf{x} + \frac{1}{2}\mathsf{x}' \in P_{t+1}$, such that equation (C.21) holds with equality. Then, by axiom A9 also equation (C.20) has to hold with equality. Equating the two equations by S'' yields the requirement

$$\begin{aligned}
& g f_t^{-1}\Big\{ \tfrac{1}{2} f_t g^{-1}\Big[(1-\beta_t)S + A\Big] + \tfrac{1}{2} f_t g^{-1}\Big[(1-\beta_t)S' + A\Big]\Big\} - A \\
&= \tfrac{(1-\beta_t)}{(1-\beta_{t+1})} g f_{t+1}^{-1}\Big\{ \tfrac{1}{2} f_{t+1} g^{-1}\Big[(1-\beta_{t+1})S\Big] + \tfrac{1}{2} f_{t+1} g^{-1}\Big[(1-\beta_{t+1})S'\Big]\Big\}
\end{aligned} \tag{C.22}$$

for all $S, S' \in [\frac{1-\beta^{T-t}}{1-\beta}\,\underline{G}, \frac{1-\beta^{T-t}}{1-\beta}\,\overline{G}]$ and $A \in [(1-\beta_t)\beta^{T-t}\,\underline{G}, (1-\beta_t)\beta^{T-t}\,\overline{G}]$.

Part II ("⇒"): In this part, I establish the general solution to equation (C.22), under the assumption that $h_t = f_t \circ g^{-1}$ is differentiable for all $t \in \{1, ..., T\}$. First, observe that the right hand side of equation (C.22) is independent of A.[13] Thus, the left hand side has to be constant in A. Taking the first derivative with respect to A, the latter requirement yields

$$\begin{aligned}
&\tfrac{\partial}{\partial A}\, h_t^{-1}\Big\{\tfrac{1}{2}h_t\,[\,(1-\beta_t)S+A]+\tfrac{1}{2}h_t\,[\,(1-\beta_t)S'+A]\,\Big\}-A=0\\
\Leftrightarrow\ & h_t^{-1\prime}\Big\{\tfrac{1}{2}h_t\,[\,(1-\beta_t)S+A]+\tfrac{1}{2}h_t\,[\,(1-\beta_t)S'+A]\,\Big\}\cdot\\
&\qquad\Big\{\tfrac{1}{2}h_t'\,[\,(1-\beta_t)S+A]+\tfrac{1}{2}h_t'\,[\,(1-\beta_t)S'+A]\,\Big\}=1\\
\Leftrightarrow\ & \tfrac{1}{2}h_t'\,[\,(1-\beta_t)S+A]+\tfrac{1}{2}h_t'\,[\,(1-\beta_t)S'+A]\\
&\qquad = h_t'\Big\{h^{-1}\Big[\tfrac{1}{2}h_t\,[\,(1-\beta_t)S+A]+\tfrac{1}{2}h_t\,[\,(1-\beta_t)S'+A]\Big]\Big\}\,,
\end{aligned}$$

where the prime at the function h_t (and only the one at the function h_t) denotes a derivative. Defining $y = h_t\,[(1-\beta_t)S+A]$ and $y' = h_t\,[(1-\beta_t)S'+A]$, both in $\mathsf{F}_t = \big(f_t(\underline{U}), f_t(\overline{U})\big)$, the latter equation becomes

$$\tfrac{1}{2}h_t' h_t^{-1}(y)+\tfrac{1}{2}h_t' h_t^{-1}(y') = h_t' h_t^{-1}\left(\tfrac{1}{2}y+\tfrac{1}{2}y'\right).$$

By Hardy et al. (1964, refinement of theorem 83 on p.74) it follows that the composition $h_t' h^{-1}$ has to be linear. Therefore, I obtain the following differential equation for h_t, where $a_t, b_t \in \mathbb{R}$ and $z = h_t^{-1}(y) \in \Gamma_t = (\underline{G}, \overline{G})$:

$$\begin{aligned}
&h_t' h_t^{-1}(y) = a_t y + b_t \qquad \forall y \in \mathsf{F}_t\\
\Leftrightarrow\quad & h_t'(z) \;=\; a_t h_t(z) + b_t \quad \forall z \in \Gamma_t.
\end{aligned} \tag{C.23}$$

For $a_t = 0$ the solution to $h_t'(z) = b_t$ is obviously $h_t(z) = b_t\, z + k_t$ with $k_t \in \mathbb{R}$. I will come back to this solution below (case 2). In the meanwhile (case 1), assume $a_t \neq 0\,\forall t \in \{1, ..., T\}$.

Case 1, $a_t \neq 0\,\forall t \in \{1, ..., T\}$:

First the differential equation (C.23) is solved using variation of the constant. Solving the homogeneous differential equation for period t yields

$$\begin{aligned}
&\textstyle\int \frac{1}{h_t}\, dh_t = \int a_t\, dz\\
\Leftrightarrow\quad & \ln h_t = a_t z + \tilde{c}_t \qquad \text{with } \tilde{c}_t \in \mathbb{R}\\
\Leftrightarrow\quad & h_t(z) = c_t \exp(a_t z) \qquad \text{with } c_t = \exp(\tilde{c}_t) \in \mathbb{R}_{++}.
\end{aligned}$$

Taking the integration constant c_t as a function of z renders the ansatz

[13] Note that a functional equation that corresponds to the requirement that the left hand side of equation (C.22) is independent of A is solved in a different way by Aczél (1966, 153) by relating it to a Cauchy equation.

$h_t(z) = c_t(z)\exp(a_t z)$ for the inhomogeneous equation:

$$\begin{array}{rcl} h_t'(z) & = & a_t h_t(z) + b_t \\ \Rightarrow c_t'(z)\exp(a_t z) + c_t(z)\,a_t\exp(a_t z) & = & a_t\,c_t(z)\exp(a_t z) + b_t \\ \Rightarrow \quad c_t'(z)\exp(a_t z) & = & b_t \\ \Rightarrow \quad \int dc_t & = & \int b_t \exp(-a_t z)dz \\ \Rightarrow \quad c_t(z) & = & -\frac{b_t}{a_t}\exp(-a_t z) + k_t \qquad \text{with } k_t \in \mathbb{R}. \end{array}$$

Therefore $h_t(z) = [-\frac{b_t}{a_t}\exp(-a_t z) + k]\exp(a_t z) = -\frac{b_t}{a_t} + k_t\,\exp(a_t z)$ *with* $k_t \in \mathbb{R}$ *is the general solution to equation* (C.23) *with* $a_t, b_t \in \mathbb{R}, a_t \neq 0$. Note, however, that it is also known by theorem 7 that h_t has to be strictly increasing. Thus, whenever for $a_t > 0$ it has to hold as well $k_t > 0$ and for $a_t < 0$ it has to hold as well $k_t < 0$. Furthermore denote $d_t = -\frac{b_t}{a_t} \in \mathbb{R}$, and determine the inverse of h_t to be $h_t^{-1}(y) = \frac{1}{a_t}\ln\left[\frac{-d_t+y}{k_t}\right]$.[14]

Second, I substitute the solution for h_t and h_{t+1} back into equation (C.22) to find for the left hand side

$$\begin{aligned} & h_t^{-1}\Big\{\tfrac{1}{2}h_t\Big[(1-\beta_t)S + A\Big] + \tfrac{1}{2}h_t\Big[(1-\beta_t)S' + A\Big]\Big\} - A \\ = & \tfrac{1}{a_t}\ln\Bigg[\tfrac{1}{k_t}\Big\{-d_t + \tfrac{1}{2}d_t + \tfrac{1}{2}k_t\exp\Big[a_t\{(1-\beta_t)S + A\}\Big] \\ & \qquad + \tfrac{1}{2}d_t + \tfrac{1}{2}k_t\exp\Big[a_t\{(1-\beta_t)S' + A\}\Big]\Big\}\Bigg] - A \\ = & \tfrac{1}{a_t}\ln\Big[\tfrac{1}{2}\exp\Big[a_t\{(1-\beta_t)S + A\}\Big] + \tfrac{1}{2}\exp\Big[a_t\{(1-\beta_t)S' + A\}\Big]\Big] - A \\ = & \tfrac{1}{a_t}\ln\Big[\tfrac{1}{2}\exp\Big[a_t\,(1-\beta_t)S\Big] + \tfrac{1}{2}\exp\Big[a_t\,(1-\beta_t)S'\Big]\Big], \\ = & \ln\Big[\tfrac{1}{2}\exp(S)^{a_t\,(1-\beta_t)} + \tfrac{1}{2}\exp(S')^{a_t\,(1-\beta_t)}\Big]^{\frac{1}{a_t}}, \end{aligned}$$

and analogously for the right hand side

$$\begin{aligned} & \tfrac{(1-\beta_t)}{(1-\beta_{t+1})}\,h_{t+1}^{-1}\Big\{\tfrac{1}{2}h_{t+1}\Big[(1-\beta_{t+1})S\Big] + \tfrac{1}{2}h_{t+1}\Big[(1-\beta_{t+1})S'\Big]\Big\} \\ = & \tfrac{(1-\beta_t)}{(1-\beta_{t+1})}\,\tfrac{1}{a_{t+1}}\ln\Big[\tfrac{1}{2}\exp\Big[a_{t+1}\,(1-\beta_{t+1})S\Big] + \tfrac{1}{2}\exp\Big[a_{t+1}\,(1-\beta_{t+1})S'\Big]\Big] \\ = & \ln\Big[\tfrac{1}{2}\exp(S)^{a_{t+1}\,(1-\beta_{t+1})} + \tfrac{1}{2}\exp(S')^{a_{t+1}\,(1-\beta_{t+1})}\Big]^{\frac{(1-\beta_t)}{a_{t+1}(1-\beta_{t+1})}}. \end{aligned} \tag{C.24}$$

Therefore, equation (C.22) requires that for a continuum of values S and S' it has to

[14] Note that the relation holds also holds $k_t < 0$. Then the nominator inside the logarithm $-d_t + y = k_t\exp(a_t z)$ is negative as well.

hold

$$\left[\tfrac{1}{2}\exp(S)^{a_t\,(1-\beta_t)} + \tfrac{1}{2}\exp(S')^{a_t\,(1-\beta_t)}\right]^{\frac{1}{a_t(1-\beta_t)}}$$
$$= \left[\tfrac{1}{2}\exp(S)^{a_{t+1}\,(1-\beta_{t+1})} + \tfrac{1}{2}\exp(S')^{a_{t+1}\,(1-\beta_{t+1})}\right]^{\frac{1}{a_{t+1}(1-\beta_{t+1})}}.$$

Necessary and sufficient for this equality is the condition $a_t(1-\beta_t) = a_{t+1}(1-\beta_{t+1}) \equiv \xi$.[15] As $a_t \in \mathbb{R}\backslash\{0\}$ and $1-\beta_t \neq 0$ for all $t \in \{1,...,T\}$, there exists a solution if and only if $\xi \in \mathbb{R}\backslash\{0\}$. Summarizing, in the case that $a_t \neq 0$ for all $t \in \{1,...,T\}$, *equation* (C.22) *implies that there exists* $\xi \in \mathbb{R}\backslash\{0\}$ *such that for every* t *it is* $h_t(z) = f_t \circ g^{-1}(z) = d_t + k_t \exp(\frac{\xi}{1-\beta_t} z)$ *with* $d_t, k_t \in \mathbb{R}, k_t \neq 0$. In addition for $\xi > 0$ it has to hold $k_t > 0$ and for $\xi < 0$ it has to hold $k_t < 0$.

Case 2, $\exists\, t \in \{1,...,T\}$ with $a_t = 0$**:**
The solution to equation (C.23) for $a_t = 0$ is $h_t(z) = b_t\, z + k_t$ with $k_t \in \mathbb{R}$. By theorem 7 it is known that h_t has to be strictly increasing. Thus, the constant b_t has to be strictly positive. But then, the constants b_t and k_t correspond to positive affine transformations of f_t, which are known not to affect the representation. Therefore, wlog I can set $b_t = 1$ and $k_t = 0$. Then h_t *is the identity* and the left hand side of equation (C.22) becomes

$$g f_t^{-1}\left\{\tfrac{1}{2} f_t g^{-1}\left[(1-\beta_t)S + A\right] + \tfrac{1}{2} f_t g^{-1}\left[(1-\beta_t)S' + A\right]\right\} - A$$
$$= \tfrac{1}{2}\left[(1-\beta_t)S + A\right] + \tfrac{1}{2}\left[(1-\beta_t)S' + A\right] - A = \tfrac{1}{2}(1-\beta_t)\left[S + S'\right]. \qquad \text{(C.25)}$$

Let me first consider the case where $a_{t+1} \neq 0$. Then, equation (C.24) gives the right hand side of equation (C.22). Define $s = \exp(S)$ and $s' = \exp(S')$ and find that equation (C.22) yields the following condition:

$$\cancel{(1-\beta_t)}\left[\tfrac{1}{2}S + \tfrac{1}{2}S'\right] = \ln\left[\tfrac{1}{2}\exp(S)^{a_{t+1}\,(1-\beta_{t+1})} + \tfrac{1}{2}\exp(S')^{a_{t+1}\,(1-\beta_{t+1})}\right]^{\frac{\cancel{(1-\beta_t)}}{a_{t+1}(1-\beta_{t+1})}}$$
$$\Leftrightarrow \quad \left[\tfrac{1}{2}\ln s + \tfrac{1}{2}\ln s'\right] = \ln\left[\tfrac{1}{2}s^{a_{t+1}\,(1-\beta_{t+1})} + \tfrac{1}{2}s^{a_{t+1}\,(1-\beta_{t+1})}\right]^{\frac{1}{a_{t+1}(1-\beta_{t+1})}}$$
$$\Leftrightarrow \quad s^{\frac{1}{2}} s'^{\frac{1}{2}} = \left[\tfrac{1}{2}s^{a_{t+1}\,(1-\beta_{t+1})} + \tfrac{1}{2}s^{a_{t+1}\,(1-\beta_{t+1})}\right]^{\frac{1}{a_{t+1}(1-\beta_{t+1})}}$$

for a continuum of s and s'. However, the above equality does not hold for $a_{t+1}(1-\beta_{t+1}) \neq 0$ (Hardy et al. 1964, 15,26). As it is $1-\beta_t \neq 0$, this fact implies a contradiction to the assumption that $a_{t+1} \neq 0$. Evaluating equation (C.22) for period $t-1$ the same reasoning brings about a contradiction to the assumption $a_{t-1} \neq 0$. Therefore, if $a_t = 0$ for some t it necessarily follows that $a_t = 0$ for all $t \in \{1,...T\}$.

In the case $a_t = 0$ for all $t \in \{1,...,T\}$ use (C.25) to see that equation (C.22) simplifies

[15] For the necessity see for example Hardy et al. (1964, 26).

to the tautology

$$\begin{array}{rcl} \frac{1}{2}\Big[(1-\beta_t)S+A\Big]+\frac{1}{2}\Big[(1-\beta_t)S'+A\Big]-A & = & \frac{(1-\beta_t)}{(1-\beta_{t+1})}\frac{1}{2}\Big[(1-\beta_{t+1})S+(1-\beta_{t+1})S'\Big] \\ \Leftrightarrow \qquad \frac{1}{2}(1-\beta_t)\Big[S+S'\Big] & = & \frac{1}{2}(1-\beta_t)\Big[S+S'\Big], \end{array}$$

which implies no further restrictions on the functional form of h_t.

Part III ("⇒"): In this part I show that the solution to equation (C.22) derived in part two has to hold as well if only continuity of $h_t = f_t \circ g^{-1}$ is assumed.[16] Other than differentiability, the latter is assured by theorem 7. Assume that some continuous function h_t satisfies equation (C.22). Expecting that the general solution will be of the form derived in part two, I define for all $t \in \{t, ..., T\}$ the continuous functions $r_t : \mathbb{R} \to \mathbb{R}$ by $r_t(y) = h_t\left[(1-\beta_t)\ln(y)\right] \Leftrightarrow h_t(z) = r_t \circ \exp(\frac{1}{1-\beta_t}z)$. Then the left hand side of equation (C.22) becomes

$$\begin{aligned} & h_t^{-1}\Big\{\tfrac{1}{2}h_t\Big[(1-\beta_t)S+A\Big]+\tfrac{1}{2}h_t\Big[(1-\beta_t)S'+A\Big]\Big\}-A \\ =\ & (1-\beta_t)\ln\circ r_t^{-1}\Big\{\tfrac{1}{2}\,r_t\circ\exp\Big[\tfrac{1}{1-\beta_t}\{(1-\beta_t)S+A\}\Big] \\ & \qquad +\tfrac{1}{2}\,r_t\circ\exp\Big[\tfrac{1}{1-\beta_t}\{(1-\beta_t)S'+A\}\Big]\Big\}-A \\ =\ & (1-\beta_t)\ln\circ r_t^{-1}\Big\{\tfrac{1}{2}\,r_t\circ\exp\Big[S+\tfrac{A}{1-\beta_t}\Big]+\tfrac{1}{2}\,r_t\circ\exp\Big[S'+\tfrac{A}{1-\beta_t}\Big]\Big\}-A \end{aligned}$$

and with defining $s = \exp[S]$, $s' = \exp[S']$ and $a = \exp\left[\frac{A}{1-\beta_t}\right]$ the relation writes as

$$\begin{aligned} &= (1-\beta_t)\ln\circ r_t^{-1}\Big\{\tfrac{1}{2}\,r_t\,(s\,a)+\tfrac{1}{2}\,r_t\,(s'\,a)\Big\}-(1-\beta_t)\ln a \\ &= (1-\beta_t)\ln\Big[\tfrac{1}{a}\,r_t^{-1}\Big\{\tfrac{1}{2}\,r_t\,(s\,a)+\tfrac{1}{2}\,r_t\,(s'\,a)\Big\}\Big]\,. \end{aligned}$$

Analogously, the right hand side of equation (C.22) becomes

$$\frac{(1-\beta_t)}{(1-\beta_{t+1})}\cdot(1-\beta_{t+1})\ln\Big[r_{t+1}^{-1}\Big\{\tfrac{1}{2}\,r_{t+1}\,(s)+\tfrac{1}{2}\,r_{t+1}\,(s')\Big\}\Big]\,.$$

Using these expressions equation (C.22) translates into the requirement

$$\frac{1}{a}\,r_t^{-1}\Big\{\tfrac{1}{2}\,r_t\,(s\,a)+\tfrac{1}{2}\,r_t\,(s'\,a)\Big\}=r_{t+1}^{-1}\Big\{\tfrac{1}{2}\,r_{t+1}\,(s)+\tfrac{1}{2}\,r_{t+1}\,(s')\Big\}$$

for a continuum of values s, s' and a. First of all, this relation implies that the left hand side has to be constant in a for all values of s and s'. By Hardy et al. (1964, 66,68) it follows that r_t *has to be either an affine transformation of* $r_t(z) = z^{\xi_t}$ *for some* $\xi_t \in \mathbb{R}\backslash\{0\}$ *or an affine transformation of* ln. I will associate the latter case with $\xi = 0$. In the first case equation (C.22) becomes

$$\frac{1}{\not a}\left\{\tfrac{1}{2}\,(s\not a)^{\xi_t}+\tfrac{1}{2}\,(s'\not a)^{\xi_t}\right\}^{\frac{1}{\xi_t}}=\left\{\tfrac{1}{2}\,(s)^{\xi_{t+1}}+\tfrac{1}{2}\,(s')^{\xi_{t+1}}\right\}^{\frac{1}{\xi_{t+1}}} \tag{C.26}$$

[16] I.e. there are no further continuous solutions to equation (C.23).

which implies $\xi_t = \xi_{t+1} \equiv \xi$ for all $t \in \{1, ..., T-1\}$ (Hardy et al. 1964, 26). The case where $r_t = \ln$ corresponds to taking the limit $\xi_t \to 0$ in (C.26), and the same reasoning on ξ_t and ξ_{t+1} holds true, i.e if some r_t is an affine transformation of ln then all have to be an affine transformation of ln.

In consequence, the following solutions of equation (C.22) for h_t are possible. In the case $\xi \in \mathbb{R}\backslash\{0\}$ I find for all $t \in \{1, ..., T-1\}$

$$h_t^*(z) = k_t \left(\exp(\tfrac{1}{1-\beta_t} z)\right)^{\xi} + d_t = k_t \exp(\tfrac{\xi}{1-\beta_t} z) + d_t \,, \tag{C.27}$$

with $d_t, k_t \in \mathbb{R}$ and, in order to assure strict increasingness of $h_t^*(z)$, $k_t\,\xi > 0$. In the case $\xi = 0$ I find for all $t \in \{1, ..., T-1\}$

$$h_t^*(z) = \tilde{b}_t \ln \left(\exp(\tfrac{1}{1-\beta_t} z)\right) + d_t = \tilde{b}_t \tfrac{1}{1-\beta_t} z + d_t \,,$$

with $\tilde{b}_t, d_t \in \mathbb{R}$ and, in order to assure strict increasingness of $h_t^*(z)$, $\tilde{b}_t\,\xi > 0$. With $\tilde{b}_t = \frac{b_t}{1-\beta_t}$ this solution is seen to correspond to case two in part two. Thus, giving up the differentiability assumption for $f_t g^{-1}$ yields no further solutions to equation (C.22), than those already found in part two.

Part IV ("⇒"): In part four I substitute the relations found in parts two and three for $h_t = f_t \circ g^{-1}$ back into the representation of $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 7. I start with the case $f_t \circ g^{-1}(y) = d_t + k_t \exp(\frac{\xi}{1-\beta_t} y)$ with $d_t, k_t \in \mathbb{R}$ and $k_t\,\xi > 0$. Taking g as given, the function f_t follow as

$$\begin{aligned} & f_t \circ g^{-1}(y) = d_t + k_t \exp(\tfrac{\xi}{1-\beta_t} y) \\ \Leftrightarrow\ & f_t(\cdot) = d_t + k_t \exp(\tfrac{\xi}{1-\beta_t} g(\cdot)) \,. \end{aligned}$$

Then the functions $\tilde{u}_t$ in the representation of theorem 7 become

$$\begin{aligned} \tilde{u}_t(x_t, p_{t+1}) &= g^{-1} \left\{ (1-\beta_t)\, g \circ u(x_t) + \beta_t\, g \circ f_{t+1}^{-1} \left[\int dp_{t+1}^{(x_{t+1}, p_{t+2})} f_{t+1} \circ \tilde{u}_{t+1} \right] \right\} \\ &= g^{-1} \Big\{ (1-\beta_t)\, g \circ u(x_t) + \beta_t\, \tfrac{1-\beta_{t+1}}{\xi} \ln \Big[\tfrac{1}{k_{t+1}} \Big\{ \\ &\qquad -d_{t+1} + \int dp_{t+1}^{(x_{t+1}, p_{t+2})} f_{t+1} \circ \tilde{u}_{t+1} \Big\} \Big] \Big\} \,. \end{aligned}$$

Define the functions $\tilde{w}_t = \frac{1}{1-\beta_t}\, g \circ \tilde{u}_t$. Due to the relation between g and f_t, imposed by axiom A9, a recursive formulation employing these strictly monotonic transformation of the functions $\tilde{u}_t$, largely simplifies the representation.

$$\begin{aligned} & \tilde{w}_T(x_T) = gu(x_T) \quad \text{and} \\ & \tilde{w}_{t-1}(x_{t-1}, p_t) = \tfrac{1}{1-\beta_{t-1}}\, g \circ \tilde{u}_{t-1}(x_{t-1}, p_t) \\ & \qquad = gu(x_{t-1}) + \tfrac{\beta_{t-1}}{\xi}\, \tfrac{1-\beta_t}{(1-\beta_{t-1})} \ln \left[\tfrac{1}{k_t} \left\{ -d_t + \int dp_t^{(x_t, p_{t+1})} f_t \tilde{u}_t \right\} \right] . \end{aligned}$$

Using the relation $\frac{1-\beta_{t+1}}{1-\beta_t} = \frac{1-\beta^{T-t+1}}{1-\beta^{T-(t+1)+1}} = \beta\beta_t^{-1}$ further yields

$$\begin{aligned}
\tilde{w}_{t-1}(x_{t-1}, p_t) &= gu(x_{t-1}) + \tfrac{\beta}{\xi}\ln\left[\tfrac{1}{k_t}\Big\{-d_t + \textstyle\int dp_t^{(x_t,p_{t+1})} f_t g^{-1}\, g\tilde{u}_t\Big\}\right] \\
&= gu(x_{t-1}) + \tfrac{\beta}{\xi}\ln\left[\tfrac{1}{k_t}\Big\{-d_t + \textstyle\int dp_t^{(x_t,p_{t+1})} d_t + k_t \exp\Big(\tfrac{\xi}{1-\beta_t}\cdot \right. \\
&\qquad\qquad \left.(1-\beta_t)\tilde{w}_t\Big)\Big\}\right] \\
&= gu(x_{t-1}) + \tfrac{\beta}{\xi}\ln\left[\textstyle\int dp_t^{(x_t,p_{t+1})} \exp\Big(\xi\tilde{w}_t\Big)\right] &\text{(C.28)} \\
&= gu(x_{t-1}) + \beta\mathcal{M}^{\exp^\xi}(p_t, \tilde{w}_t)\,, &\text{(C.29)}
\end{aligned}$$

where the uncertainty aggregation rule is characterized by the function $r(z) = \exp(\xi z) = \exp(z)^\xi$. Expression (C.29) will be used for the $g^+ - gauge$ of the representation in corollary 8, where the range of g has been fixed. Here however, the parameter ξ can be absorbed into the function g. To this end, define $\tilde{w}_t^* = |\xi|\,\tilde{w}_t$, $g^* = |\xi|\,g$ and $\text{sgn}(\xi)$ as the sign of ξ. Then line (C.28) yields

$$\begin{aligned}
\tilde{w}^*_{t-1}(x_{t-1}, p_t) &= |\xi|\, gu(x_{t-1}) + \tfrac{|\xi|}{\xi}\beta\ln\left[\textstyle\int dp_t^{(x_t,p_{t+1})} \exp\Big(\xi\tilde{w}_t\Big)\right] \\
&= g^*u(x_{t-1}) + \text{sgn}(\xi)\;\beta\ln\left[\textstyle\int dp_t^{(x_t,p_{t+1})} \exp\Big(\text{sgn}(\xi)\;\tilde{w}_t^*\Big)\right] \\
&= g^*u(x_{t-1}) + \beta\mathcal{M}^{\exp^{\text{sgn}(\xi)}}(p_t, \tilde{w}_t^*)\,. &\text{(C.30)}
\end{aligned}$$

Expression (C.30) yields equation (9.9) for the cases $f \in \{\exp, \frac{1}{\exp}\}$. To obtain the representing equation (9.10) first observe that

$$\begin{aligned}
\mathcal{M}^{f_t}(p_t, \tilde{u}_t) &= f_t^{-1}\Big[\textstyle\int dp_t^{(x_t,p_{t+1})} f_t \circ \tilde{u}_t\Big] \\
&= f_t^{-1}\Big[\textstyle\int dp_t^{(x_t,p_{t+1})} f_t g^{-1}\Big((1-\beta_t)\,\tilde{w}_t\Big)\Big] \\
&= f_t^{-1}\Big[\textstyle\int dp_t^{(x_t,p_{t+1})} d_t + k_t \exp\Big(\tfrac{\xi}{1-\beta_t}(1-\beta_t)\,\tilde{w}_t\Big)\Big] \\
&= f_t^{-1}\Big[\textstyle\int dp_t^{(x_t,p_{t+1})} d_t + k_t \exp\Big(\text{sgn}(\xi)\;\tilde{w}_t^*\Big)\Big]\,.
\end{aligned}$$

Then, recalling that $\text{sgn}(k_t) = \text{sgn}(\xi)$, find that the strictly increasing transformation

$$\begin{aligned}
\mathcal{M}^f(p_t, \tilde{w}_t^*) &= \ln\Big[\text{sgn}(\xi)\textstyle\int dp_t^{(x_t,p_{t+1})} \exp\Big(\text{sgn}(\xi)\;\tilde{w}_t^*\Big)\Big] \\
&= \mathcal{M}^{\exp^{\text{sgn}(\xi)}}\Big(p_t, \tilde{w}_t^*\Big)\,.
\end{aligned}$$

yields the expression representing the preferences in equation (9.10).

In the remaining case it is $f_t \circ g^{-1}(y) = b_t\, z + k_t$ with $b_t, k_t \in \mathrm{I\!R}$ and $b_t > 0$. Taking g as and, thus, $f_t = b_t\, z + k_t$ an analogous reasoning to the one carried out above yields

$$\begin{aligned}
\tilde{u}_t(x_t, p_{t+1}) &= g^{-1}\Big\{(1-\beta_t)\, g \circ u(x_t) + \beta_t\, g \circ f_{t+1}^{-1}\Big[\textstyle\int dp_{t+1}^{(x_{t+1},p_{t+2})} f_{t+1} \circ \tilde{u}_{t+1}\Big]\Big\} \\
&= g^{-1}\Big\{(1-\beta_t) g \circ u(x_t) + \beta_t\Big[\tfrac{1}{b_{t+1}}\Big\{-k_{t+1} + \textstyle\int dp_{t+1}^{(x_{t+1},p_{t+2})} f_{t+1} \circ \tilde{u}_{t+1}\Big\}\Big]\Big\}.
\end{aligned}$$

And defining the functions

$$\begin{aligned}
\tilde{w}_T(x_T) &= gu(x_T) \quad \text{and} \\
\tilde{w}_{t-1}(x_{t-1}, p_t) &= \tfrac{1}{1-\beta_{t-1}}\, g \circ \tilde{u}_{t-1}(x_{t-1}, p_t) \\
&= gu(x_{t-1}) + \tfrac{\beta_{t-1}}{(1-\beta_{t-1})} \Big[\tfrac{1}{b_t} \Big\{ -k_t + \textstyle\int dp_t^{(x_t,p_{t+1})} f_t \tilde{u}_t \Big\} \Big] \\
&= gu(x_{t-1}) + \tfrac{\beta_{t-1}}{(1-\beta_{t-1})} \Big[\tfrac{1}{b_t} \Big\{ -k_t + \textstyle\int dp_t^{(x_t,p_{t+1})} b_t (1-\beta_t) \tilde{w}_t + k_t \Big\} \Big] \\
&= gu(x_{t-1}) + \beta \Big[\textstyle\int dp_t^{(x_t,p_{t+1})} \tilde{w}_t \Big] \,,
\end{aligned}$$

where the latter expression corresponds to the recursion (9.9) stated in the theorem for the cases $f = \text{id}$. The representing equation (9.10) follows from

$$\begin{aligned}
\mathcal{M}^{f_t}(p_t, \tilde{u}_t) &= f_t^{-1} \Big[\textstyle\int dp_t^{(x_t,p_{t+1})} f_t \circ \tilde{u}_t \Big] \\
&= f_t^{-1} \Big[\textstyle\int dp_t^{(x_t,p_{t+1})} f_t g^{-1} \Big((1-\beta_t)\, \tilde{w}_t \Big) \Big] \\
&= f_t^{-1} \Big[\textstyle\int dp_t^{(x_t,p_{t+1})} k_t + b_t (1-\beta_t)\, \tilde{w}_t \Big]
\end{aligned}$$

which is a strictly increasing transformation of

$$\mathcal{M}^f(p_t, \tilde{w}_t) = \mathrm{E}_{p_t} \tilde{w}_t = \mathcal{M}^{\text{id}}(p_t, \tilde{w}_t) \,.$$

Part V ("$\Leftarrow$"): As shown above, the representation is a special case of theorem 7. Therefore axioms A1-A5 follow immediately from "$\Leftarrow$" of theorem 7. The following calculation shows that axiom A9 is satisfied as well. Hereto note that for certain consumption paths $\mathsf{x} \in \mathsf{X}^t$ it is $\tilde{w}_t(\mathsf{x}) = \sum_{\tau=t}^{T} \beta^{\tau-t} g \circ u(\mathsf{x}_\tau)$. For the case $h = \exp$ define $k = 1$ and for the case $h = \frac{1}{\exp}$ define $k = -1$. Then, for $h \in \{\exp, \frac{1}{\exp}\}$ and for all $t \in \{1, ..., T-1\}$, $x^0 \in X$ and $\mathsf{x}, \mathsf{x}', \mathsf{x}'' \in \mathsf{X}^{t+1}$ it holds

$$\begin{aligned}
& \tfrac{1}{2}(\mathsf{x}, x^0) + \tfrac{1}{2}(\mathsf{x}', x^0) \ \succeq_t \ (\mathsf{x}'', x^0) \\
\Leftrightarrow\ & k \ \ln \Bigg(\tfrac{1}{2} \exp \Big[k \textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} g \circ u(\mathsf{x}_{\tau+1}) \Big] \exp \Big[k\, \beta^T g \circ u(x^0) \Big] \\
& \qquad + \tfrac{1}{2} \exp \Big[k \textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} g \circ u(\mathsf{x}'_{\tau+1}) \Big] \exp \Big[k\, \beta^T g \circ u(x^0) \Big] \Bigg) \\
& \qquad\qquad \geq \textstyle\sum_{\tau=t}^{T-1} \beta^{\tau-t} g \circ u(\mathsf{x}''_{\tau+1}) + \beta^T g \circ u(x^0) \\
\Leftrightarrow\ & k \ \ln \Bigg(\tfrac{1}{2} \exp \Big[k \textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} g \circ u(\mathsf{x}_\tau) \Big] + \tfrac{1}{2} \exp \Big[k \textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} g \circ u(\mathsf{x}'_\tau) \Big] \Bigg) \\
& \qquad\qquad \geq \textstyle\sum_{\tau=t+1}^{T} \beta^{\tau-(t+1)} g \circ u(\mathsf{x}''_\tau) \\
\Leftrightarrow\ & \tfrac{1}{2} \tilde{v}_{t+1}(\mathsf{x}) + \tfrac{1}{2} \tilde{v}_{t+1}(\mathsf{x}') \geq \tilde{v}_{t+1}(\mathsf{x}'') \\
\Leftrightarrow\ & \qquad \tfrac{1}{2}\mathsf{x} + \tfrac{1}{2}\mathsf{x}' \qquad \succeq_{t+1} \mathsf{x}'' \,.
\end{aligned}$$

The case $h = \text{id}$ makes both sides of the above inequalities linear in the term $\beta^T g \circ u(x^0)$, so that it cancels as well and A9 is satisfied.

Moreover part: "$\Rightarrow$": Assume that g and g' both represent the sequence of preference relations $\succeq = (\succeq_t)_{t\in\{1,\ldots,T\}}$ (the prime in g' does *not* indicate a derivative!). By the representation of $\succeq = (\succeq_t)_{t\in\{1,\ldots,T\}}$ on certain paths, the freedom of g is limited to positive affine transformations as in theorem 7, i.e. it have to exist $a, b \in \mathbb{R}, a > 0$ such that $g = ag' + b$. However, the dependence of f_t on g destroys part of this freedom when considering choice over lotteries.[17] Precisely, find that the function $\tilde{w}'_T$ corresponding to the choice g' is

$$\tilde{w}'_T(x_T) = g' \circ u(x_T) = a\, g \circ u(x_T) + b$$

Define again $k = 1$ for the case $h = \exp$ and $k = -1$ for the case $h = \frac{1}{\exp}$. Then, for the case $h \in \{\exp, \frac{1}{\exp}\}$, the fact that $\tilde{w}_T(x_T)$ as well as $\tilde{w}'_T(x_T)$ are to represent the same preferences over period T lotteries implies

$$\begin{array}{rcl}
k \int dp_T \exp(k\,\tilde{w}_T) & \geq & k \int dp_T \exp(k\,\tilde{w}_T) \\
\Leftrightarrow k \ln \left[\int dp_T \exp(k\,\tilde{w}_T) \right] & \geq & k \ln \left[\int dp_T \exp(k\,\tilde{w}_T) \right] \\
\Leftrightarrow \quad \mathcal{M}^h(p_T, \tilde{w}_T) & \geq & \mathcal{M}^h(p'_T, \tilde{w}_T) \\
\Leftrightarrow \quad p_T & \succeq_t & p'_T \\
\Leftrightarrow \quad \mathcal{M}^h(p_T, \tilde{w}'_T) & \geq & \mathcal{M}^h(p'_T, \tilde{w}'_T) \\
\Leftrightarrow k \ln \left[\int dp_T \exp(k\,\tilde{w}'_T) \right] & \geq & k \ln \left[\int dp_T \exp(k\,\tilde{w}'_T) \right] \\
\Leftrightarrow \quad k \int dp_T \exp(k\,\tilde{w}'_T) & \geq & k \int dp_T \exp(k\,\tilde{w}'_T)
\end{array}$$

for all $p_t, p'_t \in P_t$. In consequence there have to exist constants $c, d \in \mathbb{R}, c > 0$ such that

$$\begin{aligned}
\exp(k\,\tilde{w}_T) &= c \exp(k\,\tilde{w}'_T) + d \\
&= c \exp(k\,a\,\tilde{w}'_T + kb) + d \\
&= c \exp(kb) \exp(k\,\tilde{w}'_T)^a + d\,.
\end{aligned}$$

Thus, defining the constant $\tilde{c} = c\exp(kb)$ and the variable $z = \exp(k\,\tilde{w}'_T(x_t))$ the relation

$$z = \tilde{c} z^a + d$$

has to hold for all $z \in [\exp(\underline{G}), \exp(\overline{G})]$. The relation can only be satisfied if the right hand side is linear and, thus, $a = 1$. In consequence, if g and g' both represent the preferences $\succeq = (\succeq_t)_{t\in\{1,\ldots,T\}}$, it has to exist $b \in \mathbb{R}$ such that $g = g' + b$.

For the case $h = \text{id}$, corresponding to a maximizer of intertemporally additive expected utility, the above reasoning yields no further restrictions on the constants a or b.

[17] Without the dependence of f on g an affine transformation $\mathfrak{a}$ of g cancels out. However, when f depends on g as in the representation of theorem 9, at the same time $f^{-1} \rightarrow \mathfrak{a} f^{-1}$, corresponding to an affine transformation of the inverse of f. Such a transformation is, in general, not compatible with the freedom in the choice of the representing functions.

In that case, if g and g' both represent the preferences $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ all that can be claimed is the existence $a, b \in \mathbb{R}, a > 0$, such that $g = ag' + b$.
"$\Leftarrow$": For the case $h \in \{\exp, \frac{1}{\exp}\}$, let $g = g' + b$ and g be part of a representation of $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$. Define as before $k = 1$ for the case $h = \exp$ and $k = -1$ for the case $h = \frac{1}{\exp}$. I claim that for every $t \in \{1, ..., T\}$ it exists $\gamma_t \in \mathbb{R}$ such that $\tilde{w}'_t = \tilde{w}_t + \gamma_t$. The proof is by backwards induction. For $t = T$ it holds

$$\tilde{w}'_T(x_T) = g' \circ u(x_T) = g \circ u(x_T) + b = \tilde{w}_T(x_T) + \gamma_T$$

with $\gamma_T = b$. The induction step from t to $t - 1$ works as follows:

$$\begin{aligned}\tilde{w}'_{t-1}(x_{t-1}, p_t) &= g'u(x_{t-1}) + \beta\mathcal{M}^{\exp^k}(p_t, \tilde{w}'_t)\\ &= gu(x_{t-1}) + b + k\,\beta\ln\Big[\textstyle\int dp_t^{(x_t,p_{t+1})}\exp\Big(k\tilde{w}_t + k\gamma_t\Big)\Big]\\ &= gu(x_{t-1}) + b + k\,\beta\ln\Big[\exp(k\gamma_t)\textstyle\int dp_t^{(x_t,p_{t+1})}\exp\Big(k\gamma_t\tilde{w}_t\Big)\Big]\\ &= gu(x_{t-1}) + k\,\beta\ln\Big[\textstyle\int dp_t^{(x_t,p_{t+1})}\exp\Big(k\gamma_t\tilde{w}_t\Big)\Big] + b + \beta\gamma_t\\ &= \tilde{w}_{t-1}(x_{t-1}, p_t) + \gamma_{t-1}, +\beta\gamma_t\end{aligned}$$

with $\gamma_{t-1} = b + \beta\gamma_t$. Next I show, that such an additive constant in $\tilde{w}_t$ cancels out in the representing equation (9.10):

$$\begin{aligned} & \mathcal{M}^h(p_t, \tilde{w}'_t) && \geq \mathcal{M}^h(p'_t, \tilde{w}'_t)\\ \Leftrightarrow\quad & \mathcal{M}^h(p_t, \tilde{w}_t + \gamma_t) && \geq \mathcal{M}^h(p'_t, \tilde{w}_t + \gamma_t)\\ \Leftrightarrow\quad & k\ln\big[\textstyle\int dp_t\exp(k\,\tilde{w}_t + \gamma_t)\big] && \geq k\ln\big[\textstyle\int dp_t\exp(k\,\tilde{w}_t + \gamma_t)\big]\\ \Leftrightarrow\quad & k\ln\big[\textstyle\int dp_t\exp(k\,\tilde{w}_t)\big] && \geq k\ln\big[\textstyle\int dp_t\exp(k\,\tilde{w}_t)\big]\\ \Leftrightarrow\quad & \mathcal{M}^h(p_t, \tilde{w}_t) && \geq \mathcal{M}^h(p'_t, \tilde{w}_t)\,.\end{aligned}$$

Thus, if g represents preferences $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 9 with $h \in \{\exp, \frac{1}{\exp}\}$, then so does $g' = g + b$.

In the case $h = \text{id}$, let $g = ag' + b$ and g be part of a representation of $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 9. I claim that for every $t \in \{1, ..., T\}$ it exists $\gamma_t \in \mathbb{R}$ such that $\tilde{w}'_t = a\tilde{w}_t + \gamma_t$. Proof is by backwards induction. For $t = T$ it holds

$$\tilde{w}'_T(x_T) = g' \circ u(x_T) = ag \circ u(x_T) + b = a\tilde{v}_T(x_T) + \gamma_T\,,$$

with $\gamma_T = b$. The induction step from t to $t - 1$ is as follows:

$$\begin{aligned}\tilde{w}'_{t-1}(x_{t-1}, p_t) &= g' \circ u(x_{t-1}) + \beta E_{p_t}\tilde{w}'_t(x_t, p_{t+1})\\ &= a\,g \circ u(x_{t-1}) + b + \beta E_{p_t}\,a\,\tilde{w}_t(x_t, p_{t+1}) + \beta\,\gamma_t\\ &= a\,\tilde{w}_{t-1}(x_{t-1}, p_t) + b + \beta\,\gamma_t\,.\end{aligned}$$

Setting $\gamma_{t-1} = b + \gamma_t$ closes the induction step. But then, the representation in equation

(9.10) stays unchanged:

$$E_{p_t}\,\tilde{w}'_t \;\geq\; E_{p'_t}\,\tilde{w}'_t \quad \Leftrightarrow \quad E_{p_t}\,a\,\tilde{w}_t + \gamma_t \;\geq\; E_{p'_t}\,a\,\tilde{w}_t + \gamma_t \quad \Leftrightarrow \quad E_{p_t}\,\tilde{w}_t \geq E_{p'_t}\,\tilde{w}_t\,.$$

□

Proof of theorem 10: The proof resembles that of theorem 6. Part one translates axiom A6$^{\text{s}}_{\text{st}}$ into the representation of theorem 7. Then I show in the second part that the equation derived in the first locally implies concavity of $f_t \circ g^{-1}$. Part three extends this result to concavity on the entire set Γ_t. The necessity of axiom A6$^{\text{s}}_{\text{st}}$ is implied by theorem 6. The difference to the proof of theorem 6, i.e. the stronger prerequisite in axiom A6$^{\text{s}}_{\text{st}}$, mainly affects the first step in part two. Subsequently the proof follows that of theorem 6 and the reader is referred to the latter.

Part I ("⇒"): In this part I translate axiom A6$^{\text{s}}_{\text{st}}$ into the representation of theorem 7. I start with the first line, i.e the premise, and use equation (C.19) to find

$$\begin{array}{ccc} \bar{\mathrm{x}}^t & \sim_t & \mathrm{x}^t \\ \Rightarrow g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}u(\bar{x})\Big] & = & g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}u(\mathrm{x}^t_\tau)\Big]\,. \end{array} \tag{C.31}$$

The existence of $\tau \in \{t, ..., T\}$ such that $[\mathrm{x}^t_\tau] \not\sim_\tau [\bar{x}]$ translates into

$$u(\mathrm{x}^t_\tau) \neq u(\bar{x}) \text{ for some } \tau \in \{t, ..., T\}. \tag{C.32}$$

The second line of axiom A6$^{\text{s}}_{\text{st}}$ becomes

$$\begin{array}{lcl} \bar{\mathrm{x}}^t & \succ_T & \sum_{i=t}^{T} \frac{1}{T-t+1}\,(\bar{\mathrm{x}}^t_{-i}, \mathrm{x}^t_i). \\ \Rightarrow\; g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}u(\bar{x})\Big] & & \\ \qquad > f_t^{-1}\Big[\sum_{i=t}^{T}\frac{1}{T-t+1}f_t g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}u\left((\bar{\mathrm{x}}^t_{-i}, \mathrm{x}^t_i)_\tau\right)\Big]\Big] & & \\ \Rightarrow f_t g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}u(\bar{x})\Big] & & \\ \qquad > \sum_{i=t}^{T}\frac{1}{T-t+1}f_t g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^{T}\beta^{\tau-t}u\left((\bar{\mathrm{x}}^t_{-i}, \mathrm{x}^t_i)_\tau\right)\Big]. & & \end{array}$$

Using equation (C.31) the left hand side can be transformed as follows:

$$
\begin{aligned}
& f_t g^{-1}\left[\tfrac{T-t}{T-t+1}\left[(1-\beta_t)\sum_{\tau=t}^T \beta^{\tau-t}u(\bar{x})\right]+\tfrac{1}{T-t+1}\left[(1-\beta_t)\sum_{\tau=t}^T \beta^{\tau-t}u(\mathsf{x}_\tau^t)\right]\right] \\
& \qquad\qquad > \quad \sum_{i=t}^T \tfrac{1}{T-t+1} f_t g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i}^t,\mathsf{x}_i^t)_\tau\right)\Big] \\
\Rightarrow\ & f_t g^{-1}\left[\tfrac{1}{T-t+1}\left[(1-\beta_t)\sum_{i=t}^T\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i}^t,\mathsf{x}_i^t)_\tau\right)\right]\right] \\
& \qquad\qquad > \quad \sum_{i=t}^T \tfrac{1}{T-t+1} f_t g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i}^t,\mathsf{x}_i^t)_\tau\right)\Big] \\
\Rightarrow\ & f_t g^{-1}\left[\sum_{i=t}^T\tfrac{1}{T-t+1}\left[(1-\beta_t)\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i}^t,\mathsf{x}_i^t)_\tau\right)\right]\right] \qquad\qquad \text{(C.33)} \\
& \qquad\qquad > \quad \sum_{i=t}^T \tfrac{1}{T-t+1} f_t g^{-1}\Big[(1-\beta_t)\sum_{\tau=t}^T \beta^{\tau-t}u\left((\bar{\mathsf{x}}_{-i}^t,\mathsf{x}_i^t)_\tau\right)\Big].
\end{aligned}
$$

Define the function $\tilde{z}:\mathsf{X}^t\to\Gamma_t$ by $\tilde{z}(\mathsf{x}^t)=(1-\beta_t)\sum_{\tau=t}^T\beta^{\tau-t}u\left(\mathsf{x}_\tau^t\right)$. Restricting the domain to those consumption paths that satisfy condition (C.32) the function is onto $\Big((1-\beta_t)\sum_{\tau=t}^T \underline{G}\,,\,(1-\beta_t)\sum_{\tau=t}^T \overline{G}\Big)=\big(\underline{G}\,,\,\overline{G}\big)=\Gamma_t$. In particular define $z_i=\tilde{z}\big((\mathsf{x}_{-i}^t\mathsf{x}_i^{\prime t})\big)$. In this notation equation (C.33) becomes

$$
f_t g^{-1}\left(\sum_{i=t}^T \tfrac{1}{T-t+1}\, z_i\right) \quad > \quad \sum_{i=t}^T \tfrac{1}{T-t+1} f_t g^{-1}(z_i). \qquad \text{(C.34)}
$$

If equation (C.34) had to hold for all $z_i\in\Gamma_t$ it would be a straight forward condition for strict convexity of $f_t\circ g^{-1}$. However axiom A6$^{\text{s}}_{\text{st}}$ does not immediately imply that the equation has to be met for every choice $(z_i)_{i\in\{t,\ldots,T\}}, z_i\in\Gamma_t$. Only for combination $(z_i)_{i\in\{t,\ldots,T\}}$ stemming from consumption paths $(\bar{\mathsf{x}}_{-i}^t,\mathsf{x}_i^t)$ for which $\mathsf{x}^t\in\mathsf{X}^t$ and $\bar{x}\in X$ satisfy the premise of axiom A6$^{\text{s}}_{\text{st}}$. In what follows I proceed to show that this restricted demand is enough to imply strict convexity of of $f_t\circ g^{-1}$ on Γ_t.

Part II ("⇒"): Let $z^o\in\Gamma_t$. In this part I show that for every such z^o there exists an open neighborhood $N_{z^o}\subset\Gamma_t$ such that equation (C.34) implies strict concavity of $f_t\circ g^{-1}$ on N_{z^o}.

In the first step I define a certain consumption path $\bar{\mathsf{x}}^t$ with $\bar{x}\in X$ such that $\tilde{z}(\bar{\mathsf{x}}^t)=z^o$. The fact $z^o\in\Gamma_t$ is equivalent to $\underline{G}<z^o<\overline{G}$. By connectedness of X and continuity of $g\circ u$ there exists an outcome $x^o\in u^{-1}\left[g^{-1}\left(z^o\right)\right]$ such that $z^o=u\circ g(x^o)$. Define $\mathsf{x}^{ot}=\bar{\mathsf{x}}^{ot}=(x^o,\ldots,x^o)$ and find that $\tilde{z}(\mathsf{x}^{ot})=z^0$. Note that the difference between the stationary and the non-stationary setting is that only in the stationary setting it is guaranteed that any $z^o\in\Gamma_t$ can be attained by evaluating a constant consumption path.

From step two on the proof (including **Part III**) follows exactly the one laid out for theorem 6 on page 219 with $G_\tau^o=z^o$ for all $\tau\in\{t,\ldots,T\}$ and $\epsilon=\min\{z^o-\underline{G},\overline{G}-z^o\}$.

Part IV ("⇐"): "⇐" is implied by theorem 6 for $\mathsf{x}^t=\bar{\mathsf{x}}^t$. □

Proof of lemma 7: The lemma is an immediate consequence of lemma 5 with the convention $g_1 \circ u_1 = g \circ u$. Then, the representing triples $(u, f_t, g)_{t\in\{1,\ldots,T\}}$ in the sense of theorems 7 correspond to the representing triples $(u_t = u, f_t, g_t = \beta^{t-1} g)_{t\in\{1,\ldots,T\}}$ in the sense of theorem 4. Therefore, imposing the unit, the zero level or the range of $g \circ u$ determines the according values for $g_t \circ u_t$ in the sense of theorem 4 for all periods. Thus, the statements in a), b), c) and d) in lemma 5 imply the assertions a), b), c) and d) in lemma 7. As theorems 8 and 9 are special cases of theorem 7, the reasoning holds true as well for representations in the sense of theorems 8 and 9. □

Proof of corollary 8: To the most part, the g^+−gauge of the representation in theorem 9 has already been derived in the proof of the latter theorem.
"⇒": Before absorbing the parameter ξ into the function g in the proof of theorem 9, the recursive construction of $\tilde{w}_t$ for the case corresponding to $h \in \left\{ \exp, \frac{1}{\exp} \right\}$ was given by equation (C.29), which states

$$\tilde{w}_{t-1}(x_{t-1}, p_t) = gu(x_{t-1}) + \beta \mathcal{M}^{\exp^\xi}(p_t, \tilde{w}_t)\,.$$

Simply defining the new utility function $u^* = g \circ u$ yields the $g = \text{id}$-gauge. Once the range of u^*, i.e. g, is fixed, a transformation absorbing the free parameter ξ into the function g, i.e. u^*, as carried out to arrive at the final representation stated in theorem 9, is no longer possible.

The representing equation (9.12) is obtained as follows. The representation that is known to hold by theorem 4 for the specifications of theorem 9 is

$$\begin{aligned}
\mathcal{M}^{f_t}(p_t, \tilde{u}_t) &= f_t^{-1}\Big[\int dp_t^{(x_t, p_{t+1})} f_t \circ \tilde{u}_t \Big] \\
&= f_t^{-1}\Big[\int dp_t^{(x_t, p_{t+1})} f_t g^{-1}\Big((1-\beta_t)\, \tilde{w}_t \Big) \Big] \\
&= f_t^{-1}\Big[\int dp_t^{(x_t, p_{t+1})} d_t + k_t \exp\Big(\xi\, \tilde{w}_t \Big) \Big]\,.
\end{aligned}$$

But, recalling that $k_t \xi > 0$, the latter expression is easily recognized as a strictly increasing transformation of

$$\mathcal{M}^{\exp^\xi}(p_t, \tilde{w}_t) = \tfrac{1}{\xi} \ln \Big[\int dp_t^{(x_t, p_{t+1})} \exp\Big(\xi\, \tilde{w}_t \Big) \Big]\,.$$

Therefore, also $\mathcal{M}^{\exp^\xi}(p_t, \tilde{w}_t)$ represents the preferences $\succeq = (\succeq_t)_{t\in\{1,\ldots,T\}}$.

For the case corresponding to $h = \text{id}$ in the representation of theorem 9, the proof of the latter theorem has derived the following representation

$$\begin{aligned}
&\tilde{w}_T(x_T) = gu(x_T) \quad \text{and} \\
&\tilde{w}_{t-1}(x_{t-1}, p_t) = gu(x_{t-1}) + \beta \mathrm{E}_{p_t} \tilde{w}_t\,,
\end{aligned}$$

with

$$p_t \succeq_t p'_t \quad \Leftrightarrow \quad \mathrm{E}_{p_t}\tilde{w}_t \geq \mathrm{E}_{p'_t}\tilde{w}_t \,.$$

Thus, with the definition $\mathcal{M}^{\exp^0}(p_t, \tilde{w}_t) = E_{p_t}\tilde{w}_t$, the claimed representation also holds for $\xi = 0$.

Finally, observe that, as stated in the text, the above definition of $\mathcal{M}^{\exp^0}$ corresponds to the limit $\xi \to 0$. To see this, simply apply l'Hospital's rule:

$$\begin{aligned}
\mathcal{M}^{\exp^0}(p_t, \tilde{w}_t) &\equiv \lim_{\xi \to 0} \mathcal{M}^{\exp^\xi}(p_t, \tilde{w}_t) \\
&= \lim_{\xi \to 0} \frac{\ln\left[\int dp_t \exp(\xi \tilde{w}_t)\right]}{\xi} \\
&= \lim_{\xi \to 0} \frac{\frac{\partial}{\partial \xi}\ln\left[\int dp_t \exp(\xi \tilde{w}_t)\right]}{\frac{\partial}{\partial \xi}\xi} \\
&= \lim_{\xi \to 0} \frac{\int dp_t \tilde{w}_t \exp(\xi \tilde{w}_t)}{\int dp_t \exp(\xi \tilde{w}_t)} \\
&= \frac{\int dp_t \tilde{w}_t}{1} = E_{p_t}\tilde{w}_t \,.
\end{aligned}$$

"$\Leftarrow$": Implied by theorem 9.

Moreover part: By lemma 7 the function $g \circ u$ in theorem 9 is uniquely determined, once its range has been fixed. As seen above, the representing utility function in the corollary corresponds to the function $u^* = g \circ u$. Thus, fixing its range determines the function uniquely. Moreover lemma 7 implies that the measures of intertemporal risk aversion are determined uniquely.

Equation (8.7) defines the measure of absolute intertemporal risk aversion in period t as the function

$$\mathrm{AIRA}_t(z) = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)} \,.$$

As derived in the proof of theorem 9, the case $\xi \neq 0$ corresponds to $f_t \circ g^{-1} = k_t \exp(\frac{\xi}{1-\beta_t} z) + d_t$, with $d_t, k_t \in \mathrm{I\!R}$ and $k_t\,\xi > 0$ (compare C.27). Then, with $g_1 = g$ and $g_t = \beta^{t-1} g$, the measure of absolute intertemporal risk aversion is calculated to

$$\begin{aligned}
\mathrm{AIRA}_t(z) &= -\frac{\frac{d^2}{dz^2} f_t \circ g^{-1}(\beta^{-t+1} z)}{\frac{d}{dz} f_t \circ g^{-1}(\beta^{-t+1} z)} = -\frac{\frac{d^2}{dz^2} k_t \exp\left(\frac{\xi}{1-\beta_t}\beta^{-t+1} z\right) + d_t}{\frac{d}{dz} k_t \exp\left(\frac{\xi}{1-\beta_t}\beta^{-t+1} z\right) + d_t} \\
&= -\frac{\left(\frac{\xi}{1-\beta_t}\beta^{-t+1}\right)^2 \exp\left(\frac{\xi}{1-\beta_t}\beta^{-t+1} z\right)}{\frac{\xi}{1-\beta_t}\beta^{-t+1} \exp\left(\frac{\xi}{1-\beta_t}\beta^{-t+1} z\right)} = -\frac{\xi}{\beta^{t-1}(1-\beta_t)} \,,
\end{aligned}$$

yielding the constant coefficient of absolute intertemporal risk aversion $\frac{-\xi}{\beta^{t-1}(1-\beta_t)}$. In the case $\xi = 0$ it as

$$\text{AIRA}_t(z) = -\frac{\frac{d^2}{dz^2} f_t \circ g^{-1}(\beta^{-t+1} z)}{\frac{d}{dz} f_t \circ g^{-1}(\beta^{-t+1} z)} = -\frac{\frac{d^2}{dz^2}\, b_t\, z + k_t}{\frac{d}{dz}\, b_t\, z + k_t} = 0\,,$$

coinciding with the general expression $\text{AIRA}_t(z) = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}$.

The measure of relative intertemporal risk aversion in period t is defined in equation (8.6) as the function

$$\text{RIRA}_t(z) = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)}\, z\ .$$

In consequence it holds $\text{RIRA}_t(z) = \text{AIRA}_t(z) \cdot z$, yielding $\text{RIRA}_t = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}\,\text{id}$. □

Proof of corollary 9: The proof is divided into two parts. The first part derives a representation triple in the sense of theorem 4, in which the functions f_t correspond to the identity, and which satisfies the requirements of corollary 9. The second part works out the corresponding representation as stated in the corollary. The necessity of the axioms is immediate by theorem 9.

Part I: First, observe that corollary 8 with Bernoulli utility u^* implies, with the definition $u = \exp(u^*) \Leftrightarrow u^* = \ln u$, the representation for the case $\xi = 0$ ($h = \text{id}$ in theorem 9). The logarithm is introduced because the representation for the case $\xi \neq 0$ fixes the measure scale for welfare to $\ln u^*$, as it will be observed in the remark at the end of this part of the proof. In the following, I work out the proof for the case where

$$h_t(z) = f_t \circ g^{-1}(z) = k_t \exp(\tfrac{\xi}{1-\beta_t} z) + d_t\,,$$

with $d_t, k_t \in \mathbb{R}$ and $k_t\,\xi > 0$, corresponding to equation (C.27) and case two of the proof of theorem 9. As I want to gauge the functions f_t to identity, I have to allow the functions g_t to vary over time. Therefore, I express the preferences $\succeq = (\succeq_t)_{t\in\{1,\ldots,T\}}$ in a representation in the sense of theorem 4. Recall, that a certainty stationary representation, as the one above, corresponds to a representation $(u, f_t, \beta^{t-1} g)$ in the sense of theorem 4. I take the functions f_t as given. Then, the requirement (C.27) for $f_t g^{-1}$ restated above implies

$$g_t = \beta^{t-1} g = \beta^{t-1} \tfrac{1-\beta_t}{\xi} \ln\left(\tfrac{1}{k_t}(f_t - d_t)\right)\ .$$

In consequence, the sequence of triples

$$\left(u\,,\ f_t\,,\ g_t = \beta^{t-1}\tfrac{1-\beta_t}{\xi} \ln\left(\tfrac{1}{k_t}(f_t - d_t)\right)\right)_{t\in\{1,\ldots,T\}}$$

represents the preferences described in theorem 9, in the sense of the non-stationary representation theorem 4. By gauge lemma 4 it is known that the same preferences are represented by the sequence of triples

$$\Big(u'_t = f_t \circ u\,,\; f'_t = f_t \circ f_t^{-1}\,,\; g'_t = \beta^{t-1}\tfrac{1-\beta_t}{\xi}\ln\Big(\tfrac{1}{k_t}(f_t \circ f_t^{-1} - d_t)\Big)\Big)_{t\in\{1,\dots,T\}}$$
$$= \Big(u'_t = f_t \circ u\,,\; f'_t = \mathrm{id}\,,\; g'_t = \beta^{t-1}\tfrac{1-\beta_t}{\xi}\ln\Big(\tfrac{1}{k_t}(\,\mathrm{id} - d_t)\Big)\Big)_{t\in\{1,\dots,T\}}\,. \tag{C.35}$$

As desired, uncertainty aggregation corresponding to the above representation is linear. However, observe that

$$u'_t = f_t \circ u = k_t \exp(\tfrac{\xi}{1-\beta_t} g \circ u) + d_t\,.$$

The relation implies that it is impossible to fix u'_t to a given range independent of ξ.[18] Therefore, define the functions

$$u^*_t = \Big(\tfrac{1}{k_t}(u'_t - d_t)\Big)^{\frac{1-\beta_t}{\xi}} = \exp(\tfrac{\xi}{1-\beta_t} g \circ u)^{\frac{1-\beta_t}{\xi}} = \exp(g \circ u) \tag{C.36}$$

Then $u^* = u^*_t$ is independent of ξ and moreover constant in time. Note also, that u^*_t is always positive. Using this definition, the representing triples (C.35) write as

$$\Big(k_t {u^*}^{\frac{\xi}{1-\beta_t}} + d_t\,,\; \mathrm{id}\,,\; \beta^{t-1}\tfrac{1-\beta_t}{\xi}\ln\Big(\tfrac{1}{k_t}(\,\mathrm{id} - d_t)\Big)\Big)_{t\in\{1,\dots,T\}}\,.$$

Finally, the moreover part of corollary 6 allows to eliminate the constants k_t and d_t from the above triples, up to the sign of k_t (choose $\mathfrak{a}^+_t = \frac{1}{k_t}(\,\mathrm{id} - d_t)$ and note that $f_t = \mathrm{id}$). I obtain the representing sequence of triples

$$\Big(u''_t = \mathrm{sgn}(\xi)\; {u^*}^{\frac{\xi}{1-\beta_t}}\,,\; f''_t = \mathrm{id}\,,\; g''_t = \beta^{t-1}\tfrac{1-\beta_t}{\xi}\ln\big(\mathrm{sgn}(\xi)\;\mathrm{id}\big)\Big)_{t\in\{1,\dots,T\}}\,. \tag{C.37}$$

The function u^* in expression (C.37) corresponds to the utility function u stated in corollary 9. For preference representations in theorem 9, Bernoulli utility lies in the class $u : X \to \mathbb{R}$. By equation (C.36), the latter class for u corresponds to functions u^* lying in the class of continuous functions from X into the positive real numbers, i.e. $u^* \in \{u^* : X \to \mathbb{R}_{++}\}$.

Remark: The requirement of corollary 9 that u, i.e. u^* in the representing triples above, is onto the interval U^* corresponds to setting the range for the measurement of welfare in period t to the range of

$$g''_t \circ u''_t = \beta^{t-1}\tfrac{1-\beta_t}{\xi}\ln\Big({u^*}^{\frac{\xi}{1-\beta_t}}\Big) = \beta^{t-1}\ln(u^*) \tag{C.38}$$

and, thus, $g_1 \circ u_1 = \ln u^*$.

Part II: In the following I calculate the representation expressed by the sequence of

[18] Or from a different perspective, $g'_t \circ u'_t = \beta^{t-1}\frac{1-\beta_t}{\xi}\ln\Big(\frac{1}{k_t}(u'_t - d_t)\Big)$ depends on ξ. Thus fixing the range of u'_t as is, would not allow for a common measure scale for welfare.

triples in (C.37). Let $U^* = \left[\underline{U}^*, \overline{U}^*\right]$. Then, observe that $\Delta G''_t = \beta^t \ln \frac{\overline{U}^*}{\underline{U}^*}$ and $\vartheta'_t = 0$. Thus, with the same definition for $\beta_t = 1 - \theta''_t$ as in theorem 7 (compare page 228), it holds

$$\begin{aligned}
\tilde{u}_{t-1}(x_{t-1}, p_t) &= {g''}^{-1}_{t-1}\left[\theta''_{t-1}\, g''_{t-1} \circ u''_{t-1}(x_{t-1}) + (1-\theta''_{t-1})\frac{\Delta G''_{t-1}}{\Delta G''_t}\, g''_t \circ \mathcal{M}^{f''_t}(p_t, \tilde{u}_t)\right] \\
&= {g''}^{-1}_{t-1}\left[(1-\beta_{t-1})\, g''_{t-1} \circ u''_{t-1}(x_{t-1}) + \beta_{t-1}\,\beta^{-1}\, g''_t \circ \mathcal{M}^{f''_t}(p_t, \tilde{u}_t)\right] \\
&= \operatorname{sgn}(\xi)\ \exp\left(\tfrac{\xi}{\beta^{t-2}(1-\beta_{t-1})}\left[(1-\beta_{t-1})\,\beta^{t-2}\,\tfrac{1-\beta_{t-1}}{\xi}\ln\Big(\operatorname{sgn}(\xi)\,\cdot\right.\right. \\
&\qquad \left.\left.\operatorname{sgn}(\xi)\ u^*(x_{t-1})^{\frac{\xi}{1-\beta_{t-1}}}\Big) + \beta_{t-1}\,\beta^{t-2}\,\tfrac{1-\beta_t}{\xi}\ln\Big(\operatorname{sgn}(\xi)\ \mathrm{E}_{p_t}\,\tilde{u}_t\Big)\right]\right) \\
&= \operatorname{sgn}(\xi)\ \exp\Big(\ln\big(u^*(x_{t-1})^{\xi}\big)\Big)\exp\Big(\beta_{t-1}\tfrac{1-\beta_t}{1-\beta_{t-1}}\ln\big(\operatorname{sgn}(\xi)\ \mathrm{E}_{p_t}\,\tilde{u}_t\big)\Big) \\
&= \operatorname{sgn}(\xi)\ u^*(x_{t-1})^{\xi}\big(\operatorname{sgn}(\xi)\ \mathrm{E}_{p_t}\,\tilde{u}_t\big)^{\beta}.
\end{aligned} \tag{C.39}$$

Where I have used the relation $\frac{1-\beta_t}{1-\beta_{t-1}} = \beta\beta^{-1}_{t-1}$ to arrive at the last line. Distinguishing the two cases where $\operatorname{sgn}(\xi) > 0$ and $\operatorname{sgn}(\xi) < 0$, equation (C.39) corresponds to the representation stated in the theorem.

Moreover part: Equation (C.38) in the remark shows that the demand of u^*, corresponding to u in the corollary, being onto the given interval U^* fixes also the range for the measurement of welfare $g'' \circ u''_t$. Therefore, the moreover part follows as in corollary 8. □

Proof of corollary 10: The representation is a simple transformation of corollary 9. "⇒": For $\xi \neq 0$ define $\tilde{v}_t : \tilde{X}_t \to \mathbb{R}$ for $t \in \{1, ..., T\}$ by $\tilde{v}_t = (\operatorname{sgn}(\xi)\ \tilde{u}_t)^{\frac{1}{\xi}}$, where $\tilde{u}_t$ defines the recursive construction of the representation in corollary 9. Then it is

$$\begin{aligned}
\tilde{v}_{t-1}(x_{t-1}, p_t) &= u(x_{t-1})\big(\mathrm{E}_{p_t}\ \operatorname{sgn}(\xi)\ \tilde{u}_t\big)^{\frac{\beta}{\xi}} \\
&= u(x_{t-1})\big(\mathrm{E}_{p_t}\ \tilde{v}_t^{\xi}\big)^{\frac{\beta}{\xi}} \\
&= u(x_{t-1})\ \big(\mathcal{M}^{\alpha=\xi}(p_t, \tilde{v}_t)\big)^{\beta}\ ,
\end{aligned}$$

yielding the stated construction of $\tilde{v}_t$. Then the representation of corollary 9 translates into

$$\begin{aligned}
& p_t && \succeq_t p'_t \\
\Leftrightarrow\ & \mathrm{E}_{p_t}\,\tilde{u}_t && \geq\ \mathrm{E}_{p'_t}\,\tilde{u}_t \\
\Leftrightarrow\ & \operatorname{sgn}(\xi)\ \mathrm{E}_{p_t}\ \operatorname{sgn}(\xi)\ \tilde{u}_t && \geq\ \operatorname{sgn}(\xi)\ \mathrm{E}_{p'_t}\ \operatorname{sgn}(\xi)\ \tilde{u}_t \\
\Leftrightarrow\ & \operatorname{sgn}(\xi)\ \mathrm{E}_{p_t}\,\tilde{v}_t^{\xi} && \geq\ \operatorname{sgn}(\xi)\ \mathrm{E}_{p'_t}\,\tilde{v}_t^{\xi}
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \quad \left(\mathrm{E}_{p_t}\,\tilde{v}_t^{\xi}\right)^{\xi} \quad \geq \left(\mathrm{E}_{p_t'}\,\tilde{v}_t^{\xi}\right)^{\xi} \\
&\Leftrightarrow \mathcal{M}^{\alpha=\xi}(p_t', \tilde{v}_t) \geq \mathcal{M}^{\alpha=\xi}(p_t', \tilde{v}_t)
\end{aligned}$$

for all $p_t, p_t' \in P_t$.

For the case $\xi = 0$ the stated representation corresponds to

$$\begin{aligned}
\mathcal{M}^0(p_t, \tilde{v}_t) &= \exp\left(\int dp_t \ln \tilde{v}_t\right) \\
&= \exp\left(\int dp_t \ln\left[u(x_t)\left(\exp\left[\mathrm{E}_{p_t}\ \ln(\tilde{v}_{t+1})\right]\right)^{\beta}\right]\right) \\
&= \exp\left(\int dp_t \ln u(x_t) + \beta\left[\mathrm{E}_{p_t}\ \ln(\tilde{v}_{t+1})\right]\right)
\end{aligned}$$

Define $u^* = \ln u$ and $\tilde{v}_t^* = \ln \tilde{v}_t$. Then the representation is ordinally equivalent to

$$\int dp_t u^*(x_t) + \beta\left[\mathrm{E}_{p_t}\ \ln(\tilde{v}_{t+1})\right]$$

For the case $\xi = 0$ the stated representation corresponds to

$$\begin{aligned}
\tilde{v}_{t-1}(x_{t-1}, p_t) &= u(x_{t-1})\left(\exp\left[\mathrm{E}_{p_t}\ \ln(\tilde{v}_t)\right]\right)^{\beta} \\
&= u(x_{t-1}) \exp\left[\mathrm{E}_{p_t}\,\beta \ln(\tilde{v}_t)\right].
\end{aligned}$$

Defining $\tilde{v}_t^* = \ln \tilde{v}_t$ and $u^* = \ln u \Leftrightarrow u = \exp u^*$ yields the representation

$$\begin{aligned}
\tilde{v}_{t-1}^*(x_{t-1}, p_t) &= \ln\left(\exp[u^*(x_{t-1})] \exp\left[\mathrm{E}_{p_t}\,\beta \tilde{v}_t^*\right]\right) \\
&= u^*(x_{t-1}) + \mathrm{E}_{p_t}\,\beta\,\tilde{v}_t^*.
\end{aligned}$$

But the latter construction of aggregate welfare, corresponds to that of corollary 8 for preferences corresponding to $\xi = 0$ (intertemporally additive expected utility). Moreover, the uncertainty evaluation

$$\begin{aligned}
\mathcal{M}^0(p_t, \tilde{v}_t) &= \exp\left(\int dp_t\ \ln \tilde{v}_t\right) \\
&= \exp\left(\int dp_t\ \tilde{v}_t^*\right)
\end{aligned}$$

is a strictly increasing transformation of $\mathrm{E}_{p_t}\tilde{v}_t^*$. Therefore, the representation for the case $\xi = 0$ is equivalent to the formulation in corollary 8.

"$\Leftarrow$": Immediate consequence of corollary 9.

Moreover part: Is implied by the moreover part of corollary 9. Again, the measure scale for welfare is fixed for the first period to the range $\ln U^*$. □

C.3 Proofs for Chapter 10

Proof of theorem 11: "⇒": Translating the condition (10.2) for a preference for an early resolution of uncertainty into the representation of theorem 4 yields the inequality

$$\begin{aligned}
&\lambda(x_t, p_{t+1}) + (1-\lambda)(x_t, p'_{t+1}) \quad \succeq_t \quad (x_t, \lambda p_{t+1} + (1-\lambda)p'_{t+1}) \\
\Leftrightarrow\ &\lambda f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1}\mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big] \\
&\quad +(1-\lambda) f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1}\mathcal{M}^{f_{t+1}}(p'_{t+1}, \tilde{u}_{t+1}) + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big] \\
&\geq f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}\Big[\lambda \int dp_{t+1} f_{t+1}\tilde{u}_{t+1} \\
&\qquad\qquad +(1-\lambda)\int dp_{t+1} f_{t+1}\tilde{u}_{t+1}\Big] + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big]
\end{aligned}$$

Defining $\gamma = \int dp_{t+1} f_{t+1}\tilde{u}_{t+1} \in f_{t+1}(U_{t+1})$ and $\gamma' = \int dp'_{t+1} f_{t+1}\tilde{u}_{t+1} \in f_{t+1}(U_{t+1})$ yields

$$\begin{aligned}
\Leftrightarrow\ &\lambda f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}(\gamma) + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big] \\
&\quad +(1-\lambda) f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}(\gamma') + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big] \\
&\geq f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}\Big[\lambda\gamma + (1-\lambda)\gamma'\Big] + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big].
\end{aligned} \tag{C.40}$$

Condition (10.2) in the definition of a preference for an early resolution of uncertainty has to hold for all lotteries $p_{t+1}, p'_{t+1} \in P_{t+1}$ and $\lambda \in [0,1]$. Therefore, inequality (C.40) has to hold for all $\gamma, \gamma' \in f_{t+1}(U_{t+1})$, implying that the expression

$$f_t \circ g_t^{-1}\left[\theta_t\, g_t \circ u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1} \circ f_{t+1}^{-1}(\gamma) + \theta_t\theta_{t+1}^{-1}\vartheta_t\right] \tag{C.41}$$

has to be convex in $\gamma \in f_{t+1}(U_{t+1})$. Similarly, replacing $\succeq_t$ by $\preceq_t$ and $\geq$ by $\leq$ for the case of a preference for a late resolution of uncertainty implies concavity of expression (C.41).
"⇐": All steps showing the necessity of the convexity condition of expression (C.41) can be carried out backwards, implying sufficiency. □

Proof of theorem 12: The proof is divided into three parts. First, I translate axiom A9 into the representation theorem 7 and derive a functional equation with known solution. In the second part I work out the corresponding representation and translate it from recursive lotteries into probabilities defined directly on consumption paths. Finally, part three shows that the functional representation implies the axioms.

Part I ("⇒"): By axioms A1-A3, A4' and A5' a representation for $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ in the sense theorem 4 has to exist. Indifference with respect to the timing of uncertainty resolution in the sense of axiom A9, formally corresponds to a weak preference for early *and* late resolution of uncertainty. Therefore, relation (C.40), worked out in the proof of theorem 11, has to hold with equality. Defining $h_t = g_t \circ f_t^{-1}\,\forall\, t \in \{1, ..., T\}$, $y = \theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\vartheta_t$, $z = \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}(\gamma)$ and $z' = \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}(\gamma')$ yields the following equality

$$\begin{aligned}
&\lambda f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}(\gamma) + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big] \\
&\quad +(1-\lambda) f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}(\gamma') + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big] \\
&= f_t g_t^{-1}\Big[\theta_t\, g_t u_t(x_t) + \theta_t\theta_{t+1}^{-1}\, g_{t+1} f_{t+1}^{-1}\Big[\lambda\gamma + (1-\lambda)\gamma'\Big] + \theta_t\theta_{t+1}^{-1}\vartheta_t\Big] \\
\Leftrightarrow\ &\lambda h_t\Big[y+z\Big] + (1-\lambda) h_t\Big[y+z'\Big] \\
&= h_t\Big[y + \theta_t\theta_{t+1}^{-1} h_{t+1}^{-1}\Big[\lambda h_{t+1}(\theta_{t+1}\theta_t^{-1} z) + (1-\lambda) h_{t+1}(\theta_{t+1}\theta_t^{-1} z')\Big]\Big] \\
\Leftrightarrow\ & h_t^{-1}\Big[\lambda h_t\Big[y+z\Big] + (1-\lambda) h_t\Big[y+z'\Big]\Big] - y \\
&= \theta_t\theta_{t+1}^{-1} h_{t+1}^{-1}\Big[\lambda h_{t+1}(\theta_{t+1}\theta_t^{-1} z) + (1-\lambda) h_{t+1}(\theta_{t+1}\theta_t^{-1} z')\Big]\,. \qquad \text{(C.42)}
\end{aligned}$$

As the right hand side of equation (C.42) does not depend on y, the left hand side has to be constant in y. By part two and part three of the proof of theorem 9, or by Aczél (1966, 153), the only solutions for h_t satisfying this condition are

$$h_t(z) = a_t \exp(\xi_t z) + b_t \qquad \text{and}$$
$$h_t(z) = a_t z + b_t$$

with $a_t, \xi_t \neq 0$.

Case 1, $h_1(z) = a_1 \exp(\xi_1 z) + b_1$**:**

In the case $h_1(z) = a_1 \exp(\xi_1 z) + b_1$ note that either a_1 and ξ_1 have to be both negative, or they have to be both positive, in order to yield an increasing function h_t (as required in theorem 4). In both cases, the inverse calculates to $h_1^{-1}(z) = \frac{1}{\xi_1}\ln(\frac{z-b_1}{a_1})$. Then the left hand side of equation (C.42) translates for $t=1$ into

$$\begin{aligned}
&\tfrac{1}{\xi_1}\ln\Big[\tfrac{1}{a_1}\Big\{\lambda\big(a_1\exp(\xi_1 y)\exp(\xi_1 z) + b_1\big) \\
&\qquad +(1-\lambda)\big(a_1\exp(\xi_1 y)\exp(\xi_1 z') + b_1\big) - b_1\Big\}\Big] - y \\
&= \tfrac{1}{\xi_1}\ln\Big[\lambda\big(\exp(\xi_1 z)\big) + (1-\lambda)\big(\exp(\xi_1 z')\big)\Big]\,.
\end{aligned}$$

Define the functions $r(z) = \exp(\xi_1\, z)$ and $s(z) = h_2(\theta_2\theta_1^{-1} z)$. Then equation (C.42) for $t=1$ turns into the relation

$$r^{-1}\Big[\lambda r(z) + (1-\lambda) r(z')\Big] = s^{-1}\Big[\lambda s(z) + (1-\lambda) s(z')\Big]\,.$$

By Hardy et al. (1964, 66) it follows that S is a nondegenerate affine transformation of r. Therefore, it must hold that $s(z) = h_2(\theta_2\theta_1^{-1} z) = a_2\exp(\xi_1\, z) + b_2$. Then,

$$h_2(z) = a_2\exp(\tfrac{\xi_1\,\theta_1}{\theta_2}\, z) + b_2\,.$$

Defining $\xi_2 = \frac{\xi_1 \theta_1}{\theta_2}$, the same reasoning for $t = 2$ yields $h_3(\theta_3 \theta_2^{-1} z) = a_3 \exp(\xi_2 z) + b_3$

$$h_3(z) = a_3 \exp(\tfrac{\xi_2 \theta_2}{\theta_3} z) + b_3 \,.$$

Defining inductively $\xi_{t+1} = \frac{\xi_t \theta_t}{\theta_{t+1}}$ and recognizing that the definition implies that $\theta_t \xi_t \equiv \xi$ is constant over time, find inductively that for all $t \in \{1, ..., T\}$:

$$h_t(z) = a_t \exp(\tfrac{\xi}{\theta_t} z) + b_t \qquad \text{(C.43)}$$

with $a_t \xi > 0$.

Case 2, $h_1(z) = a_1 z + b_1$**:**

For h_1 linear, the same reasoning as in case 1 implies that h_t has to be linear for all periods. Thus, the solution in case 2 is

$$h_t(z) = a_t z + b_t$$

for all $t \in \{1, ..., T\}$. Moreover, the constants a_t have to be strictly positive in order to yield an increasing function h_t as required by theorem 4.

Part II ("⇒"): In part two I derive the representations corresponding to the functions $h_t = f_t \circ g_t^{-1}$ derived in part one. Starting with **case 1**, I have $h_t = a_t \exp(\frac{\xi}{\theta_t} z) + b_t$ for all $t \in \{1, ..., T\}$. Defining the affine function $\mathfrak{a}_t(z) = a_t z + b_t$ simplifies the notation to $h_t(z) = \mathfrak{a}_t \exp(\frac{\xi}{\theta_t} z)$ and $h_t^{-1}(z) = \frac{\theta_t}{\xi} \ln(\mathfrak{a}_t^{-1} z)$ for the inverse. To explore this relation it proves helpful to employ a strictly monotonic transformation of the functions $\tilde{u}_t$ for the recursive construction of the representation. Defining these as $\tilde{v}_t = f_t \circ \tilde{u}_t$ for all $\{1, ..., T\}$, find find

$$\begin{aligned}
\tilde{v}_t(x_t, p_{t+1}) &= f_t \circ \tilde{u}_t(x_t, p_{t+1}) \\
&= h_t\Big[\theta_t g_t u_t(x_t) + \theta_t \theta_{t+1}^{-1} h_{t+1}^{-1} \left[\textstyle\int dp_{t+1} f_{t+1} \tilde{u}_{t+1}\right] + \theta_t \theta_{t+1}^{-1} \vartheta_t\Big] \\
&= \mathfrak{a}_t \exp\left(\tfrac{\xi}{\theta_t}\Big[\theta_t g_t u_t(x_t) + \theta_t \theta_{t+1}^{-1} \tfrac{\theta_{t+1}}{\xi} \ln\left[\mathfrak{a}_{t+1}^{-1} \textstyle\int dp_{t+1} \tilde{v}_{t+1}\right] + \theta_t \theta_{t+1}^{-1} \vartheta_t\Big]\right) \\
&= \mathfrak{a}_t \exp\left(\xi g_t u_t(x_t) + \ln\left[\mathfrak{a}_{t+1}^{-1} \textstyle\int dp_{t+1} \tilde{v}_{t+1}\right] + \xi \theta_{t+1}^{-1} \vartheta_t\right) \\
&= \mathfrak{a}_t \exp(\xi g_t u_t(x_t)) \left[\mathfrak{a}_{t+1}^{-1} \textstyle\int dp_{t+1} \tilde{v}_{t+1}\right] \exp(\xi \theta_{t+1}^{-1} \vartheta_t) \,.
\end{aligned}$$

Recursively this relation yields

$$\begin{aligned}
\tilde{v}_t(x_t, p_{t+1}) &= f_t \circ \tilde{u}_t(x_t, p_{t+1}) \\
&= \mathfrak{a}_t \exp(\xi g_t u_t(x_t)) \Big[\mathfrak{a}_{t+1}^{-1} \int dp_{t+1} \mathfrak{a}_{t+1} \exp(\xi g_{t+1} u_{t+1}(x_{t+1})) \\
&\qquad \big[\mathfrak{a}_{t+2}^{-1} \int dp_{t+2} \tilde{v}_{t+2} \big] \exp(\xi \theta_{t+2}^{-1} \vartheta_{t+1}) \Big] \exp(\xi \theta_{t+1}^{-1} \vartheta_t) \\
&= \mathfrak{a}_t \exp(\xi g_t u_t(x_t)) \Big[\int dp_{t+1} \exp(\xi g_{t+1} u_{t+1}(x_{t+1})) \\
&\qquad \int dp_{t+2} \mathfrak{a}_{t+2}^{-1} \tilde{v}_{t+2} \Big] \exp(\xi \theta_{t+2}^{-1} \vartheta_{t+1}) \exp(\xi \theta_{t+1}^{-1} \vartheta_t) \\
&= \mathfrak{a}_t \exp(\xi g_t u_t(x_t)) \Big[\prod_{\tau=t+1}^{T} \int dp_\tau \exp(\xi g_\tau u_\tau(x_\tau)) \Big] \\
&\qquad \prod_{\tau=t}^{T-1} \exp(\xi \theta_{\tau+1}^{-1} \vartheta_\tau) \,.
\end{aligned}$$

Observe that the brackets delimit the argument of the product, not of the integrals. The integral $\int dp_\tau^{(x_\tau, p_{\tau+1})}$ is not only over the argument x_τ of the expression $\exp(\xi g_\tau u_\tau(x_\tau))$, but also over the measure $p_{\tau+1}$ determining the integration in the subsequent period. Then, preferences in period t are represented by the uncertainty aggregation rule

$$\begin{aligned}
\mathcal{M}^{f_t}(p_t, \tilde{u}_t) &= f_t^{-1} \Big[\int dp_t^{(x_t, p_{t+1})} f_t \circ \tilde{u}_t \Big] = f_t^{-1} \Big[\int dp_t^{(x_t, p_{t+1})} \tilde{v}_t \Big] \,. \\
&= f_t^{-1} \Big[\int dp_t^{(x_t, p_{t+1})} \mathfrak{a}_t \exp(\xi g_t u_t(x_t)) \Big[\prod_{\tau=t+1}^{T} \int dp_\tau \exp(\xi g_\tau u_\tau(x_\tau)) \Big] \\
&\qquad \prod_{\tau=t}^{T-1} \exp(\xi \theta_{\tau+1}^{-1} \vartheta_\tau) \Big] \\
&= f_t^{-1} \Big[\mathfrak{a}_t \Big[\prod_{\tau=t}^{T} \int dp_\tau \exp(\xi g_\tau u_\tau(x_\tau)) \Big] \prod_{\tau=t}^{T-1} \exp(\xi \theta_{\tau+1}^{-1} \vartheta_\tau) \Big]
\end{aligned}$$

With $\mathcal{M}^{f_t}(p_t, \tilde{u}_t)$ being a representation, so is any strictly increasing transformation. In particular, note that f_t^{-1} is strictly increasing and that $\prod_{\tau=t}^{T-1} \exp(\xi \theta_{\tau+1}^{-1} \vartheta_\tau)$ is a positive constant. Furthermore, let $k = 1$ for $a_t, \xi > 0$, and $k = -1$ for $a_t, \xi < 0$. Then, the following expression represents preferences over uncertain period t lotteries expressed in terms of the recursive lotteries p_t

$$k \prod_{\tau=t}^{T} \int dp_\tau^{(x_\tau, p_{\tau+1})} \exp(\xi g_\tau u_\tau(x_\tau)) \,, \tag{C.44}$$

in the sense that

$$p_t \succeq_t p_t' \;\Leftrightarrow\; k \prod_{\tau=t}^{T} \int dp_\tau \exp(\xi g_\tau u_\tau(x_\tau)) \;\geq\; k \prod_{\tau=t}^{T} \int dp_\tau' \exp(\xi g_\tau u_\tau(x_\tau))$$

for all $p_t, p_t' \in P_t$.

Finally, I translate the representation (C.44) into the representation stated in the theorem applying the reduced probability measures p_t^{X} defined on the set of consumption paths. In chapter 10.2 I have shown how to infer such a probability measure $p^{\mathsf{X}_t} \in \Delta(\mathsf{X}^t)$ from any given temporal lottery $p_t \in P_t, t \in \{1, ..., T\}$. The probability measures defined in that section will now be used to rearrange the expression (C.44) into the representation stated in the theorem. To start with, I demonstrate the according transformation of the first two measures p_t and p_{t+1} in expression (C.44). To this purpose, I suppress the terms that that go beyond period $\tau = t+2$.

$$\begin{aligned}&\int_{X_t\times P_{t+1}} dp_t^{(x_t,p_{t+1})} \exp(\xi g_t u_t(x_t)) \int_{X_{t+1}\times P_{t+2}} dp_{t+1}^{(x_{t+1},p_{t+2})} \exp(\xi g_{t+1} u_t(x_{t+1}))\\ &= \int_{X_t} d\mathbb{P}^{X_t}(x_t) \int_{P_{t+1}} d\mathbb{P}^{P_{t+1}|x_t}(p_{t+1}) \int_{X_{t+1}\times P_{t+2}} dp_{t+1}^{(x_{t+1},p_{t+2})}(x_{t+1},p_{t+2})\\ &\quad \exp(\xi g_t u_t(x_t)) \exp(\xi g_{t+1} u_t(x_{t+1}))\,.\end{aligned}$$

Observe that $\mathbb{P}^{P_{t+1}|x_t}(p_{t+1})$ is a function of x_t. Moreover, no term in the expression directly depends on the measure p_{t+1} in the third integral (only on its arguments). 'Integrating out' p_{t+1} eliminates the temporal information on what uncertainty resolves in period t, and yields the following expression (see chapter 10.2):

$$\begin{aligned}&= \int_{X_t} d\mathbb{P}^{X_t}(x_t) \int_{X_{t+1}\times P_{t+2}} d\mathbb{P}^{X_{t+1},P_{t+2}|x_t}(x_{t+1},p_{t+2})\\ &\quad \exp(\xi g_t u_t(x_t)) \exp(\xi g_{t+1} u_t(x_{t+1}))\,.\end{aligned}$$

Inductively this manipulation can be carried on for all periods $\tau \in \{t+1,...T\}$ as follows:

$$\begin{aligned}&\int_{X_\tau\times P_{\tau+1}} d\mathbb{P}^{X_\tau,P_{\tau+1}|x_{\tau-1},...,x_t}(x_\tau,p_{\tau+1}) \int_{X_{\tau+1}\times P_{\tau+2}} dp_{\tau+1}^{(x_{\tau+1},p_{\tau+2})}\\ &= \int_{X_\tau\times P_{\tau+1}} d\mathbb{P}^{X_\tau|x_{\tau-1},...,x_t}(x_\tau)\, d\mathbb{P}^{P_{\tau+1}|x_\tau,...,x_t}(p_{\tau+1}) \int_{X_{\tau+1}\times P_{\tau+2}} dp_{\tau+1}^{(x_{\tau+1},p_{\tau+2})}\\ &= \int_{X_\tau} d\mathbb{P}^{X_\tau|x_{\tau-1},...,x_t}(x_\tau) \int_{X_{\tau+1}\times P_{\tau+2}} d\mathbb{P}^{X_{\tau+1},P_{\tau+2}|x_\tau,...,x_t}(x_{\tau+1},p_{\tau+2})\,,\end{aligned}$$

where, in general, the measure $\mathbb{P}^{X_{\tau+1},P_{\tau+2}|x_\tau,...,x_t}$ depends on the integration variables $x_\tau,...,x_t$ of the preceding integrations. See chapter 10.2 for details. Applying this manipulation to all periods transforms the representation (C.44) into

$$\begin{aligned}&k \prod_{\tau=t}^T \int_{X_\tau} d\mathbb{P}^{X_\tau|x_{\tau-1},...,x_t}(x_\tau)\ \exp(\xi g_\tau u_\tau(x_\tau))\\ &= k \int_{\mathsf{X}^t} dp_t^{\mathsf{X}}\ \prod_{\tau=t}^T \exp(\xi g_\tau u_\tau(x_\tau))\\ &= k \int_{\mathsf{X}^t} dp_t^{\mathsf{X}}\ \exp\left(\sum_{\tau=t}^T \xi g_\tau u_\tau(x_\tau)\right)\,, \qquad \text{(C.45)}\end{aligned}$$

with $k<0$ for $\xi<0$ and $k>0$ for $\xi>0$. Define, as in equation (10.6) stated in the theorem, the aggregate utility function $\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}_\tau^t)$. Replacing the variable k in expression (C.45) with the function $\frac{1}{\xi}\ln$, corresponding to a strictly increasing transformation, yields the representation

$$p_t \succeq_t p'_t \Leftrightarrow \mathcal{M}^{\exp^\xi}(p_t^{\mathsf{X}},\tilde{u}_t) \geq \mathcal{M}^{\exp^\xi}(p_t'^{\mathsf{X}},\tilde{u}_t) \quad \forall\, p_t,p'_t \in P_t\,. \qquad \text{(C.46)}$$

It would correspond to a u^+−gauge, when the range of the functions g_t had been fixed exogeneously. However, as the latter is not the case in theorem 12, I can redefine the function g_t as $g_t^* = |\xi|\, g_t$ to absorb the constant ξ up to its sign. Then, For $\xi>0$ define $h=\exp$ to obtain the representation stated in equation (10.7). For $\xi<0$, defining $h = \exp^{-1} = \frac{1}{\exp}$ takes up the sign of ξ and, thus, also yields the preference representation stated in equation (10.7).

In **case 2**, it was found that $h_t = a_t\, z + b_t$ for all $t \in \{1, ..., T\}$. Again, the definition of the affine functions $\mathfrak{a}_t(z) = a_t\, z + b_t$, simplifies the notation to $h_t(z) = \mathfrak{a}_t\, z$ and $h_t^{-1}(z) = \mathfrak{a}_t^{-1}\, z$. Then, I find

$$\begin{aligned}
\tilde{v}_t(x_t, p_{t+1}) &= f_t \circ \tilde{u}_t(x_t, p_{t+1}) \\
&= h_t\Big[\theta_t g_t u_t(x_t) + \theta_t \theta_{t+1}^{-1} h_{t+1}^{-1} \left[\textstyle\int dp_{t+1} f_{t+1} \tilde{u}_{t+1}\right] + \theta_t \theta_{t+1}^{-1} \vartheta_t\Big] \\
&= \mathfrak{a}_t\Big[\theta_t g_t u_t(x_t) + \theta_t \theta_{t+1}^{-1} \left[\textstyle\int dp_{t+1} \mathfrak{a}_{t+1}^{-1} \tilde{v}_{t+1}\right] + \theta_t \theta_{t+1}^{-1} \vartheta_t\Big] ,
\end{aligned}$$

which recursively yields

$$\begin{aligned}
\tilde{v}_t(x_t, p_{t+1}) = {}& \mathfrak{a}_t\Big[\theta_t g_t u_t(x_t) + \theta_t \theta_{t+1}^{-1} \Big[\textstyle\int dp_{t+1} \mathfrak{a}_{t+1}^{-1} \mathfrak{a}_{t+1} \Big[\theta_{t+1} g_{t+1} u_{t+1}(x_{t+1}) + \\
& \qquad \theta_{t+1} \theta_{t+2}^{-1} \Big[\textstyle\int dp_{t+2} \mathfrak{a}_{t+2}^{-1} \tilde{v}_{t+2}\Big] + \theta_{t+1} \theta_{t+2}^{-1} \vartheta_{t+1}\Big]\Big] + \theta_t \theta_{t+1}^{-1} \vartheta_t\Big] \\
= {}& \mathfrak{a}_t \theta_t \Big[g_t u_t(x_t) + \Big[\textstyle\int dp_{t+1} \Big[g_{t+1} u_{t+1}(x_{t+1}) + \\
& \qquad \Big[\textstyle\int dp_{t+2} \theta_{t+2}^{-1} \mathfrak{a}_{t+2}^{-1} \tilde{v}_{t+2}\Big] + \theta_{t+2}^{-1} \vartheta_{t+1}\Big]\Big] + \theta_{t+1}^{-1} \vartheta_t\Big] \\
= {}& \mathfrak{a}_t \theta_t \Big[g_t u_t(x_t) + \textstyle\int dp_{t+1} g_{t+1} u_{t+1}(x_{t+1}) + \int dp_{t+2} \theta_{t+2}^{-1} \mathfrak{a}_{t+2}^{-1} \tilde{v}_{t+2} \\
& \qquad + \theta_{t+2}^{-1} \vartheta_{t+1} + \theta_{t+1}^{-1} \vartheta_t\Big] \\
= {}& \mathfrak{a}_t \theta_t \Big[g_t u_t(x_t) + \textstyle\sum_{\tau=t+1}^{T} \int dp_\tau g_\tau u_\tau(x_\tau)\Big] + \mathfrak{a}_t \theta_t \sum_{\tau=t}^{T-1} \theta_{\tau+1}^{-1} \vartheta_\tau\, .
\end{aligned}$$

Again, the integral $\int dp_\tau^{(x_\tau, p_{\tau+1})}$ is not only over x_τ, but also over the measure $p_{\tau+1}$, determining the integration in the subsequent period. Preferences in period t are represented by the uncertainty aggregation rule

$$\begin{aligned}
\mathcal{M}^{f_t}(p_t, \tilde{u}_t) &= f_t^{-1}\Big[\textstyle\int dp_t^{(x_t, p_{t+1})} f_t \circ \tilde{u}_t\Big] = f_t^{-1}\Big[\textstyle\int dp_t^{(x_t, p_{t+1})} \tilde{v}_t\Big]\, . \\
&= f_t^{-1}\Big[\mathfrak{a}_t \theta_t \Big[\textstyle\sum_{\tau=t}^{T} \int dp_\tau g_\tau u_\tau(x_\tau)\Big] + \mathfrak{a}_t \theta_t \sum_{\tau=t}^{T-1} \theta_{\tau+1}^{-1} \vartheta_\tau\Big]\, .
\end{aligned}$$

With $\mathcal{M}^{f_t}(p_t, \tilde{u}_t)$, also any strictly increasing transformation represents period t preferences. In particular, the following expression constitutes a preference representation

$$\textstyle\sum_{\tau=t}^{T} \int dp_\tau g_\tau u_\tau(x_\tau)\, ,$$

so that

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \textstyle\sum_{\tau=t}^{T} \int dp_\tau g_\tau u_\tau(x_\tau) \;\geq\; \sum_{\tau=t}^{T} \int dp'_\tau g_\tau u_\tau(x_\tau)\, ,$$

for all $p_t, p'_t \in P_t$.

The translation into the non-recursive representation works similar to case 1,

$$\begin{aligned}
&\sum_{\tau=t}^{T} \int dp_\tau g_\tau u_\tau(x_\tau) \\
&= \int dp_t^{x_t,p_{t+1}} \left[g_t u_t(x_t) + \int dp_{t+1}^{x_{t+1},p_{t+2}} \left[g_{t+1} u_{t+1}(x_{t+1}) + \left[... + \int dp_T^{x_T} g_T u_T(x_T) \right] \right] \right] \\
&= \int dp_t^{x_t,p_{t+1}} \int dp_{t+1}^{x_{t+1},p_{t+2}} ... \int dp_T^{x_T} g_T u_t(x_T) \\
&\quad\ g_t u_t(x_t) + g_{t+1} u_{t+1}(x_{t+1}) + ... + g_T u_T(x_T) \\
&= \int d\mathbb{P}^{X_t}(x_t) d\mathbb{P}^{X_{t+1}|x_t}(x_{t+1}) \,....\, d\mathbb{P}^{X_T|x_{T-1},...,x_t}(x_T) \\
&\quad\ g_t u_t(x_t) + g_{t+1} u_{t+1}(x_{t+1}) + ... + g_T u_T(x_T) \\
&= \int dp_t^{\mathsf{x}} \sum_{\tau=t}^{T} g_\tau u_\tau(x_\tau) \\
&= \int dp_t^{\mathsf{x}} \tilde{u}_t(\mathsf{x}^t)\,,
\end{aligned}$$

yielding the representation stated in equation (10.7) for $h = \text{id}$.

Part III ("$\Leftarrow$"): As shown above, the representation is a special case of theorem 4. Therefore axioms A1-A5' follow immediately from "$\Leftarrow$" of theorem 4. The following calculation shows that axiom A10 is satisfied as well. For the case $h = \exp$ define $k = 1$ and for the case $h = \frac{1}{\exp}$ define $k = -1$. Moreover, let $p_t^e = \lambda(x_t, p_{t+1}) + (1-\lambda)(x_t, p'_{t+1})$ denote the lottery with early resolution of uncertainty and let $p_t^l = (x_t, \lambda p_{t+1} + (1-\lambda) p'_{t+1})$ denote the lottery with late resolution of uncertainty. Then, for $h \in \{\exp, \frac{1}{\exp}\}$ and for all $t \in \{1, ..., T-1\}$, $x_t \in X$, $p_{t+1}, p'_{t+1} \in P_{t+1}$ and $\lambda \in [0,1]$ it holds

$$\begin{aligned}
\mathcal{M}^h\left(p_t^{e\mathsf{x}}, \tilde{u}_t\right) &= k \ln \left[\int dp_t^{e\mathsf{x}} \exp\left(\sum_{\tau=t}^{T} k g_\tau u_\tau(x_\tau) \right) \right] \\
&= k \ln \Big[\lambda \exp\left(k g_t u_t(x_t)\right) \int dp_{t+1}^{\mathsf{x}} \exp\left(\sum_{\tau=t+1}^{T} k g_\tau u_\tau(x_\tau) \right) \\
&\qquad + (1-\lambda) \exp\left(k g_t u_t(x_t)\right) \int dp_{t+1}^{\prime\mathsf{x}} \exp\left(\sum_{\tau=t+1}^{T} k g_\tau u_\tau(x_\tau) \right) \Big] \\
&= k \ln \Big[\exp\left(k g_t u_t(x_t)\right) \Big(\lambda \int dp_{t+1}^{\mathsf{x}} + (1-\lambda) \int dp_{t+1}^{\prime\mathsf{x}} \\
&\qquad \exp\left(\sum_{\tau=t+1}^{T} k g_\tau u_\tau(x_\tau) \right) \Big] \\
&= k \ln \left[\int dp_t^{l\mathsf{x}} \exp\left(\sum_{\tau=t}^{T} k g_\tau u_\tau(x_\tau) \right) \right] \\
&= \mathcal{M}^h\left(p_t^{l\mathsf{x}}, \tilde{u}_t\right)
\end{aligned}$$

Thus indifference between early and late resolution of uncertainty prevails. In the case $h = \text{id}$ the uncertainty aggregation rules are linear and the same equality in the representation holds.

Moreover part: "$\Rightarrow$": That two different sequences $(g_t)_{t\in\{1,...,T\}}$ and $(g'_t)_{t\in\{1,...,T\}}$ representing $(\succeq_t)_{t\in\{1,...,T\}}$ at most differ up to positive affine transformations follows from the representation on certain consumption path and the moreover part in theorem 4. The proof that the common multiplicative factor a in $g'_t = a g_t + b_t$ has to equal unity for the cases $h \in \{\exp, \frac{1}{\exp}\}$ is the same as in the proof of the moreover part of theorem 9.

"$\Leftarrow$": Let $g'_t = g_t + b_t$ with $b_t \in \mathbb{R}$ for all $t \in \{1, ..., T\}$. Define as before $k = 1$ for the case $h = \exp$ and $k = -1$ for the case $h = \frac{1}{\exp}$. Then, for the case $h \in \{\exp, \frac{1}{\exp}\}$, the following equality holds for all $t \in \{1, ..., T\}$.

$$\begin{aligned}\mathcal{M}^h\left(p_t^{\mathsf{X}}, \tilde{u}'_t\right) &= k \ln\left[\int dp_t^{\mathsf{X}}\ \exp\left(\sum_{\tau=t}^T k g_\tau u_\tau(x_\tau) + k b_\tau\right)\right] \\ &= k \ln\left[\int dp_t^{\mathsf{X}}\ \exp\left(\sum_{\tau=t}^T k g_\tau u_\tau(x_\tau)\right)\right] + \sum_{\tau=t}^T b_\tau \\ &= \mathcal{M}^h\left(p_t^{\mathsf{X}}, \tilde{u}_t\right) + \sum_{\tau=t}^T b_\tau\end{aligned}$$

Thus, if one of the sequences $(g_t)_{t\in\{1,...,T\}}$ and $(g'_t)_{t\in\{1,...,T\}}$ represents $(\succeq_t)_{t\in\{1,...,T\}}$, so does the other:

$$\begin{aligned} & \mathcal{M}^h\left(p_t^{\mathsf{X}}, \tilde{u}'_t\right) && \geq \mathcal{M}^h\left(p_t'^{\mathsf{X}}, \tilde{u}'_t\right) \\ \Leftrightarrow\ & \mathcal{M}^h\left(p_t^{\mathsf{X}}, \tilde{u}_t\right) + \sum_{\tau=t}^T b_\tau && \geq \mathcal{M}^h\left(p_t'^{\mathsf{X}}, \tilde{u}_t\right) + \sum_{\tau=t}^T b_\tau \\ \Leftrightarrow\ & \mathcal{M}^h\left(p_t^{\mathsf{X}}, \tilde{u}_t\right) && \geq \mathcal{M}^h\left(p_t'^{\mathsf{X}}, \tilde{u}_t\right)\ .\end{aligned}$$

For the case where $h = \mathrm{id}$, intertemporal and uncertainty aggregation are linear and the equivalence

$$\begin{aligned} & \mathrm{E}_{p_t^{\mathsf{X}}}\ \tilde{u}'_t(\mathsf{x}^t) && \geq \mathrm{E}_{p_t'^{\mathsf{X}}}\ \tilde{u}'_t(\mathsf{x}^t) \\ \Leftrightarrow\ & \mathrm{E}_{p_t^{\mathsf{X}}}\ a\,\tilde{u}_t(\mathsf{x}^t) + \sum_{\tau=t}^T b_\tau && \geq \mathrm{E}_{p_t'^{\mathsf{X}}}\ a\,\tilde{u}_t\,(\mathsf{x}^t) + \sum_{\tau=t}^T b_\tau \\ \Leftrightarrow\ & \mathrm{E}_{p_t^{\mathsf{X}}}\ \tilde{u}'_t(\mathsf{x}^t) && \geq \mathrm{E}_{p_t'^{\mathsf{X}}}\ \tilde{u}'_t(\mathsf{x}^t)\end{aligned}$$

implies that $(g_t)_{t\in\{1,...,T\}}$ and $(g'_t)_{t\in\{1,...,T\}}$ both represent $(\succeq_t)_{t\in\{1,...,T\}}$ if the relation $g'_t = a g_t + b_t$ holds with $a \in \mathbb{R}_{++}$ and $b_\tau \in \mathbb{R}$ for all $t \in \{1, ..., T\}$. □

Proof of corollary 11: Using the representation of corollary 7 instead of theorem 4, when calculating the representation in the proof of theorem 12, allows to trade in the freedom to pick the functions u_t in order to set the functions g_t to identity. In consequence, the aggregate utility function is defined as $\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^T u_\tau(\mathsf{x}^t_\tau)$. Moreover, with $g_t = \mathrm{id}$ for all $t \in \{1, ..., T\}$, fixing u_t also fixes $g_t \circ u_t$. Therefore, lemma 5 assures that the risk measure AIRA_t and RIRA_t are unique. Precisely, fixing welfare for the worst outcomes in all periods in the best outcome in one period is an immediate alternative to cases c) and d) in lemma 5 to eliminate the freedom in the choice of g_t for all $t \in \{1, ..., T\}$. Then, the representation (C.46) derived in the proof of theorem 4 holds:

$$p_t \succeq_t p'_t \ \Leftrightarrow\ \mathcal{M}^{\exp^\xi}(p_t^{\mathsf{X}}, \tilde{u}_t) \geq \mathcal{M}^{\exp^\xi}(p_t'^{\mathsf{X}}, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t\,,$$

with $\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^T u_\tau(\mathsf{x}^t_\tau)$ and without the freedom to absorb ξ into the functions $g_t \circ u_t = u_t$, as done in the proof of theorem 4. The proof that the case where $h = \mathrm{id}$

corresponds to the (limiting) definition of $\mathcal{M}^{\exp^0}(p_t^{\mathsf{X}}, \tilde{u}_t) = \mathrm{E}_{p_t^{\mathsf{X}}}\, \tilde{u}_t$ is found in the proof of corollary 8.

Moreover part: As noted above, lemma 7 implies the uniqueness of the risk measures AIRA_t and RIRA_t. Equation (8.7) defines the measure of absolute intertemporal risk aversion in period t as the function

$$\mathrm{AIRA}_t(z) = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)} .$$

As derived in the proof of theorem 12, the case $\xi \neq 0$ corresponds to

$$h_t(z) = g_t \circ f_t^{-1}(z) = a_t \exp(\tfrac{\xi}{\theta_t}\, z) + b_t$$

Therefore, the measure of absolute intertemporal risk aversion calculates to

$$\begin{aligned} \mathrm{AIRA}_t(z) &= -\frac{\frac{d^2}{dz^2} f_t \circ g_t^{-1}(z)}{\frac{d}{dz} f_t \circ g_t^{-1}(z)} = -\frac{\frac{d^2}{dz^2} k_t \exp\left(\frac{\xi}{\theta_t} z\right) + d_t}{\frac{d}{dz} k_t \exp\left(\frac{\xi}{\theta_t} z\right) + d_t} \\ &= -\frac{\left(\frac{\xi}{\theta_t}\right)^2 \exp\left(\frac{\xi}{\theta_t} z\right)}{\frac{\xi}{\theta_t} \exp\left(\frac{\xi}{\theta_t} z\right)} = -\frac{\xi}{\theta_t} , \end{aligned}$$

for $\xi \neq 0$. The same relation is easily seen to hold as well for $\xi = 0$ where $f_t \circ g_t$ is linear in z. The measure of relative intertemporal risk aversion in period t is defined in equation (8.6) as the function

$$\mathrm{RIRA}_t(z) = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)}\, z .$$

In consequence it holds $\mathrm{RIRA}_t(z) = \mathrm{AIRA}_t(z) \cdot z$, yielding $\mathrm{RIRA}_t = -\frac{\xi}{\theta_t}\,\mathrm{id}$. □

Proof of theorem 13: "⇒": Adding certainty stationarity to the assumptions of theorem 12 implies, as shown in the proof of theorem 7, that Bernoulli utility can be picked identical in all periods. Moreover, in that case it exist $\beta \in \mathrm{I\!R}_{++}$ and $g : \mathsf{X} \to \mathrm{I\!R}$ such that the functions g_t can be chosen as $g_t = \beta^{t-1} g$. Then, in the representation of theorem 12, the construction of aggregate utility simplifies to the form

$$\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^{T} g_\tau \circ u_\tau(\mathsf{x}_\tau^t) = \sum_{\tau=t}^{T} \beta^{\tau-1} g \circ u(\mathsf{x}_\tau^t) \equiv \sum_{\tau=t}^{T} \beta^{\tau-1} u^*(\mathsf{x}_\tau^t) , \tag{C.47}$$

where the simple redefinition of Bernoulli utility as $u^*(\mathsf{x}_\tau^t) = g \circ u(\mathsf{x}_\tau^t)$ yields the $g = \mathrm{id}-$gauge. Moreover, in the formulation of the theorem, the range of welfare $u^*(\mathsf{x}_\tau^t) = g \circ u(\mathsf{x}_\tau^t)$, i.e. u in the notation of the theorem, is fixed exogenously. Therefore, as in corollary 11, the parameter ξ in equation (C.46) stemming from the relation

$$h_t(z) = g_t \circ f_t^{-1}(z) = a_t \exp(\tfrac{\xi}{1-\beta_t}\, z) + b_t \tag{C.48}$$

cannot be absorbed by the function u^*. In consequence, for the case $\xi \neq 0$, the representation (C.46) prevails, just as for the g^+-gauge in corollary 11. Also as in the previous g^+- corollaries 8 and 11, it is found that the case $\xi = 0$ is covered by the representation using the uncertainty aggregation rule $\mathcal{M}^{\exp^0}(p_t^{\mathsf{X}}, \tilde{u}_t) = \mathrm{E}_{p_t^{\mathsf{X}}}\, \tilde{u}_t$.
"$\Leftarrow$": Implied by theorems 7 and 12.
Moreover part: By lemma 7, the choice of the range of $u = u_1 = g_1 \circ u_1$ as W^* fixes the measure scale of welfare for all periods. Therefore, corollary 11 covers the moreover part with $\theta_t = 1 - \beta_t$ (see proof of theorem 7). □

Proof of corollary 12: "$\Rightarrow$": First, gauge lemma 4 is applied to translate the sequence of representing triples from the general representation into the Kreps Porteus gauge. Then, the (per period) utility function employed in corollary 12 is chosen in order to be time invariant and fix the measure scale for welfare. Finally, instead of working out the representation from the beginning, the new functions u and g_t are substituted for the old ones into the reduced non-recursive representation.

The preferences in the general certainty stationary representation corresponding to the form given in equation (C.47) for the case $\xi \neq 0$ correspond to the sequence of representing triples

$$\Big(u\,,\; k_t \exp(\tfrac{\xi}{1-\beta_t} g_t) + d_t\,,\; g_t\Big)_{t\in\{1,\ldots,T\}} \,. \tag{C.49}$$

An application of gauge lemma 5 with $s = k_t \exp(\frac{\xi}{1-\beta_t} g_t) + d_t$ renders uncertainty aggregation linear. It transforms the representation to

$$\Big(k_t \exp(\tfrac{\xi}{1-\beta_t} g_t \circ u) + d_t\,,\; \mathrm{id}\,,\; \tfrac{1-\beta_t}{\xi} \ln\Big(\tfrac{1}{k_t}(\mathrm{id} - d_t)\Big)\Big)_{t\in\{1,\ldots,T\}}$$

The moreover part of corollary 6 allows to eliminate the constants k_t and d_t from the above triples, up to the sign of k_t (choose $\mathfrak{a}_t^+ = \frac{1}{k_t}(\,\mathrm{id} - d_t)$ and note that $f_t = \mathrm{id}$). Furthermore, it is $g_t = \beta^{t-1} g$, implying the representation

$$\Big(u' = \mathrm{sgn}(\xi)\; \exp\big(\tfrac{\xi}{1-\beta_t}\beta^{t-1} g \circ u\big)\,,\; f_t' = \mathrm{id}\,,\; g_t' = \tfrac{1-\beta_t}{\xi} \ln\big(\mathrm{sgn}(\xi)\; \mathrm{id}\big)\Big)_{t\in\{1,\ldots,T\}} \,.$$

In the representation of corollary 12, I use a particular utility function u, henceforth u^*, whose range can be fixed independent of the period, and which determines the ranges of welfare $g' \circ u'$ independent of ξ. Define

$$u^* = \big(\mathrm{sgn}(\xi)\; u'\big)^{\frac{1-\beta_t}{\xi\,\beta^{t-1}}} = \Big(\mathrm{sgn}(\xi)\; \mathrm{sgn}(\xi)\; \exp\big(\tfrac{\xi}{1-\beta_t}\beta^{t-1} g \circ u\big)\Big)^{\frac{1-\beta_t}{\xi\,\beta^{t-1}}} = \exp\big(g \circ u\big) \,.$$

Then, the function u^* is independent of t (as g and u are so), it is always positive (being an image of the exponential function) and, moreover, it fixes welfare independent of the parameter ξ, as can be inferred from the following equation:

$$g' \circ u' = \tfrac{1-\beta_t}{\xi} \ln\big(\operatorname{sgn}(\xi)\ u'\big) = \tfrac{1-\beta_t}{\xi} \ln\left(\operatorname{sgn}(\xi)\ \operatorname{sgn}(\xi)\ u^{*\,\frac{\xi\,\beta^{t-1}}{1-\beta_t}}\right) = \beta^{t-1} \ln u^* .$$

To obtain the corresponding representation observe that $g_t = \beta^{t-1} g$ in the representation corresponding to the triples (C.49) has to be transformed with the inverse of $s = \exp(\frac{\xi}{1-\beta_t}\beta^{t-1}g)$ to $\frac{1-\beta_t}{\xi} \ln\big(\operatorname{sgn}(\xi)\operatorname{id}\big)$ while f becomes the identity and u is replaced with $u' = \operatorname{sgn}(\xi)\big(u^*\big)^{\frac{\xi\,\beta^{t-1}}{1-\beta_t}}$. Applying these changes to the representation (C.45) yields

$$\begin{aligned}
k \textstyle\int dp_t^{\mathsf{X}}\ \exp\Big(\sum_{\tau=t}^T \xi g_\tau u_\tau(x_\tau)\Big) &= \operatorname{sgn}(\xi) \textstyle\int dp_t^{\mathsf{X}}\ \exp\Big(\sum_{\tau=t}^T \xi\beta^{\tau-1} g u(x_\tau)\Big) \\
&= \operatorname{sgn}(\xi) \textstyle\int dp_t^{\mathsf{X}}\ \exp\left(\sum_{\tau=t}^T \xi \frac{1-\beta_t}{\xi} \ln\left(\operatorname{sgn}(\xi)\ \operatorname{sgn}(\xi)\big(u^*(x_\tau)\big)^{\frac{\xi\,\beta^{t-1}}{1-\beta_t}}\right)\right) \\
&= \operatorname{sgn}(\xi) \textstyle\int dp_t^{\mathsf{X}}\ \exp\Big(\sum_{\tau=t}^T \ln\big(u^*(x_\tau)\big)^{\xi\,\beta^{t-1}}\Big) \\
&= \operatorname{sgn}(\xi) \textstyle\int dp_t^{\mathsf{X}}\ \prod_{\tau=t}^T \big(u^*(x_\tau)\big)^{\xi\,\beta^{t-1}} .
\end{aligned} \tag{C.50}$$

Defining $\tilde{u} = \operatorname{sgn}(\xi) \prod_{\tau=t}^T \big(u'(x_\tau)\big)^{\xi\,\beta^{t-1}}$ brings about the representation stated in corollary 12 for the case $\xi \neq 0$. For the case $\xi = 0$ the representation equals that of theorem 13 as $\mathcal{M}^{\exp^0}(p_t^{\mathsf{X}}, \tilde{u}_t) = \mathrm{E}_{p_t^{\mathsf{X}}}\ \tilde{u}_t$ (see corollary 8).
"$\Leftarrow$": Since the representation is shown to be equivalent to that of theorem 13, necessity of the axioms follows from the proof of the latter theorem.
Moreover part: As the the coefficients of intertemporal risk aversion are gague invariant, the result is covered by corollary 8. □

Proof of corollary 13: "$\Rightarrow$": The corollary is an immediate consequence of the representation established in equation (C.50) in the proof of corollary 12, which, with $u \equiv u^*$, is a strictly increasing transformation of the expression

$$\operatorname{sgn}(\xi)\left(\textstyle\int dp_t^{\mathsf{X}}\ \prod_{\tau=t}^T \big(u(x_\tau)\big)^{\xi\,\beta^{t-1}}\right)^{\frac{1}{|\xi|}} .$$

The latter is again ordinally equivalent to

$$\left(\textstyle\int dp_t^{\mathsf{X}}\ \prod_{\tau=t}^T \big(u(x_\tau)\big)^{\xi\,\beta^{t-1}}\right)^{\frac{1}{\xi}} .$$

Defining $\tilde{u} = \prod_{\tau=t}^T \big(u'(x_\tau)\big)^{\beta^{t-1}}$ yields the representation stated in corollary 13 for the case $\xi \neq 0$. For the case $\xi = 0$ the stated representation is

$$\begin{aligned} \mathcal{M}^{\xi=0}(p_t^{\mathsf{X}}, \tilde{u}_t) &= \exp\left(\int dp_t^{\mathsf{X}} \ \ln\left(\prod_{\tau=t}^T \left(u(x_\tau)\right)^{\beta^{t-1}}\right)\right) \\ &= \exp\left(\int dp_t^{\mathsf{X}} \ \sum_{\tau=t}^T \ln\left(u(x_\tau)\right)^{\beta^{t-1}}\right) \\ &= \exp\left(\int dp_t^{\mathsf{X}} \ \sum_{\tau=t}^T \beta^{t-1} \ln\left(u(x_\tau)\right)\right) . \end{aligned} \tag{C.51}$$

Thus, Bernoulli utility used in the above representation is a logarithmic transformation of the certainty additive Bernoulli utility function (welfare) employed in the representation of theorem 13. The *class* of preferences represented by the evaluation corresponding to expression (C.51) is obviously the same as those represented for $\xi = 0$ in theorem 13 or corollary 12.

"$\Leftarrow$": Implied by corollary 12.

Moreover part: Implied by corollary 12. □

Proof of corollary 14: "$\Rightarrow$": I give the reasoning as well for the stationary as for the non-stationary setting, as the latter is referred to in the text. For the non-stationary setting equation (C.46) in the proof of theorem 12 has already pointed out the representation for the general u^+ gauge as

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{\exp^\xi}(p_t^{\mathsf{X}}, \tilde{u}_t) \geq \mathcal{M}^{\exp^\xi}(p_t'^{\mathsf{X}}, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t\,,$$

where aggregate welfare has been defined through the recursion

$$\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}_\tau^t)\,.$$

For the Epstein-Zin gauge, the outcome space is assumed to be a subset of $\mathbb{R}$. Moreover Bernoulli utility (in every period) is assumed to be strictly increasing in the consumption level. Thus the identity can be chosen as the representing Bernoulli utility function. Then, with $u = u_t = \text{id}$, in the non-stationary case aggregate utility is characterized by

$$\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^T g_\tau(\mathsf{x}_\tau^t)\,, \tag{C.52}$$

and the form of the evaluation of uncertain scenarios, characterized by intertemporal risk aversion, stays unchanged. For the certainty stationary setting it is $g_t = \beta^{t-1} g$ and thus

$$\tilde{u}_t(\mathsf{x}^t) = \sum_{\tau=t}^T \beta^{t-1} g(\mathsf{x}_\tau^t)\,.$$

"$\Leftarrow$": Implied by theorem 13.

Moreover part: See corollary 11 (non-stationary setting) and theorem 13 (certainty stationary setting).

□

Proof of theorem 14: The assertion follows immediately from comparing the functions characterizing intertemporal risk aversion in the representations of theorem 9 and theorem 13. These imply that the two representations can only coincide for the case where $\beta = 1$.
"$\Rightarrow$": Preferences satisfying the stated axioms have to be representable in the sense of theorems 9 and 13.[19] Choose a nondegenerate closed interval $W^* \subset \mathrm{I\!R}_{++}$ and require that $u = u^{\text{welf}}$ is onto W^*. Then, due to risk stationarity, by corollary 8 there have to exist ξ and β such that the functions $f_t \circ g_t$ characterizing intertemporal risk aversion are specified by the coefficients $\text{AIRA}_t = -\frac{\xi}{\beta^{t-1}(1-\beta_t)}$. Analogously, due to timing indifference, by theorem 13 there have to exist ξ' and β' such that the functions $f_t \circ g_t$ characterizing intertemporal risk aversion are specified by the coefficients $\text{AIRA}_t = -\frac{\xi'}{1-\beta'_t}$.

Both representations, that of corollary 8 and that of theorem 13, are special cases of the certainty stationary representation in theorem 7. For given preferences $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$, coincidence of the representations on certain consumption paths implies that $\beta = \beta'$. In consequence, it also holds that $\beta_t = \beta'_t$. As the measure scale for welfare is fixed to W^* in the first period, lemma 7 states that the characterizations AIRA_t of intertemporal risk aversion are unique for all $t \in \{1, ..., T\}$. Therefore, comparison of the measures of intertemporal risk aversion for period one implies that $\xi = \xi'$. Then, the requirement that furthermore $\text{AIRA}_t \stackrel{!}{=} -\frac{\xi}{\beta^{t-1}(1-\beta_t)} \stackrel{!}{=} -\frac{\xi}{(1-\beta_t)}$ for all $t > 1$, cannot be satisfied unless $\beta = 1$ or $\xi = 0$. However, the requirement of strict intertemporal risk aversion as formulated in axiom $\text{A6}^{\text{s}}_{\text{st}}$ implies implies $\xi < 0$. Therefore it has to hold that $\beta = 1$.
"$\Leftarrow$": Except for axiom $\text{A6}^{\text{s}}_{\text{st}}$ all of the stated axioms are implied by theorems 9 and 13. Axiom $\text{A6}^{\text{s}}_{\text{st}}$ is implied by theorem 10, case a). □

[19]Recall that axiom A9 implies certainty stationarity as described in axiom A7.

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